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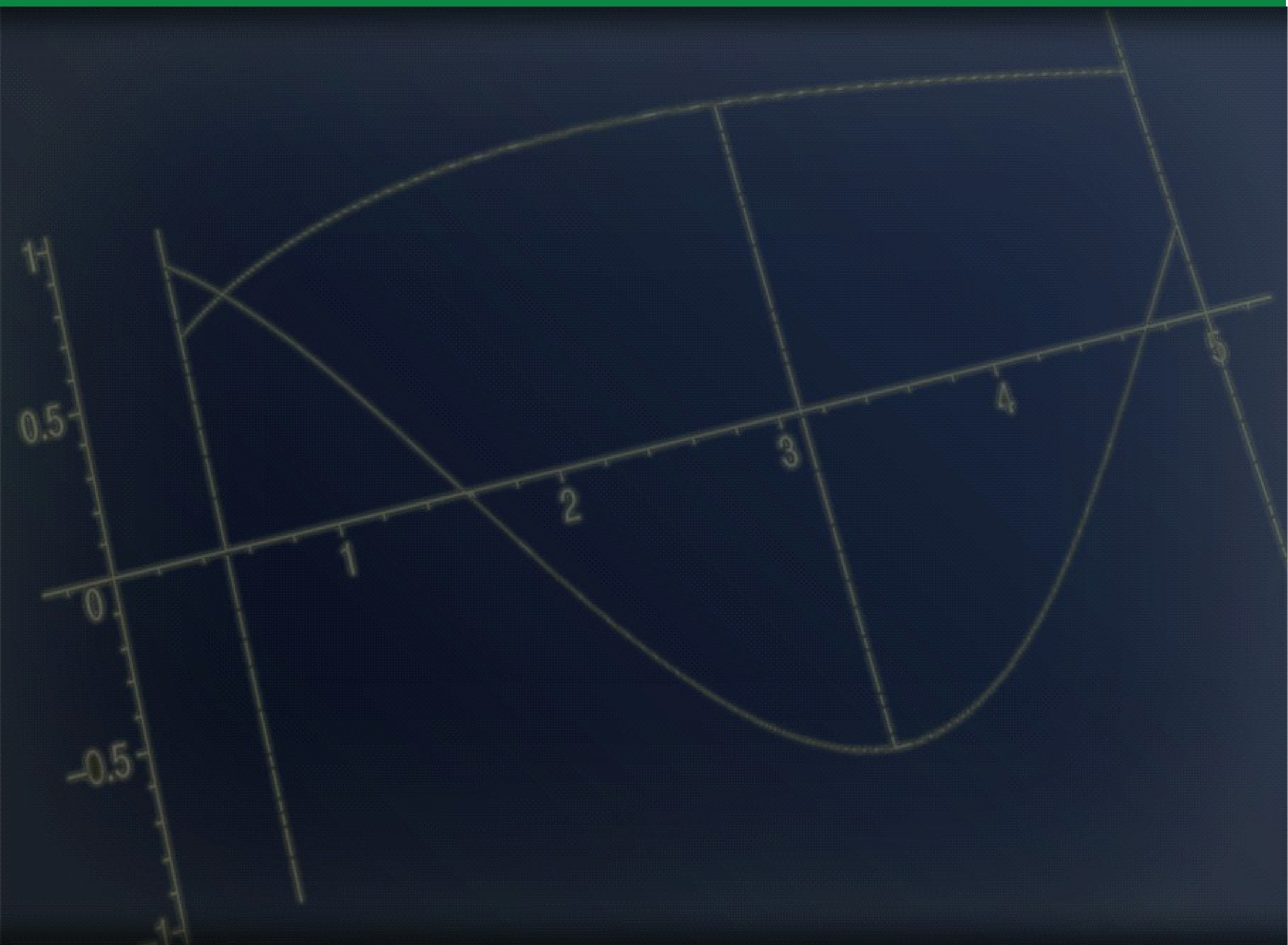
GLOBAL ANALYSIS

FUNCTIONAL ANALYSIS EXAMPLES C-1



Leif Mejlbro

Global Analysis

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1 Metric Spaces

Example 1.1 *Let (X, d_X) and (Y, d_Y) be metric spaces. Define*

$$d_{X\times Y} : (X \times Y) \times (X \times Y) \to \mathbb{R}_0^+$$

by

$$d_{X\times Y}\left((x_1, y_1), (x_2, y_2)\right) = \max\left(d_X(x_1, x_2), d_Y(y_1, y_2)\right).$$

1. *Show that $d_{X\times Y}$ is a metric on $X \times Y$.*
2. *Show that the projections*

$$p_X : X \times Y \to X, \qquad p_X(x, y) = x,$$

$$p_Y : X \times Y \to Y, \qquad p_Y(x, y) = y,$$

are continuous mappings.

The geometric interpretation is that $d_{X\times Y}$ compares the distances of the coordinates and then chooses the largest of them.

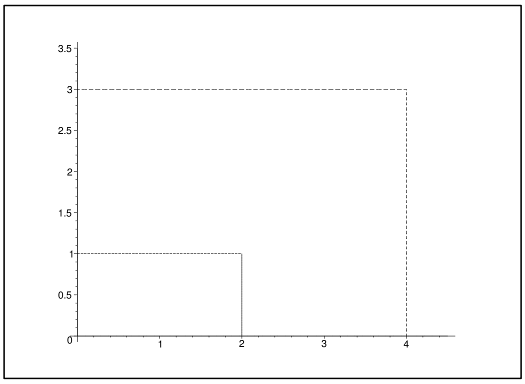


Figure 1: The points (x_1, y_1) and (x_2, y_2), and their projections onto the two coordinate axes.

1. **MET 1.** We have assumed that d_X and d_Y are metrics, hence

$$d_{X\times Y}\left((x_1, y_1), (x_2, y_2)\right) = \max\left(d_X(x_1, x_2), d_Y(y_1, y_2)\right) \geq \max(0, 0) = 0.$$

If

$$d_{X\times Y}\left((x_1, y_1), (x_2, y_2)\right) = \max\left(d_X(x_1, x_2), d_Y(y_1, y_2)\right) = 0,$$

then

$$d_X(x_1, y_1) = 0 \qquad \text{and} \qquad d_Y(y_1, y_2) = 0.$$

Using that d_X and d_Y are metrics, this implies by **MET 1** for d_X and d_Y that $x_1 = x_2$ and $y_1 = y_2$, thus

$$(x_1, y_1) = (x_2, y_2),$$

and **MET 1** is proved for $d_{X\times Y}$.

MET 2. From d_X and d_Y being symmetric it follows that

$$\begin{aligned} d_{X\times Y}\left((x_1, y_1), (x_2, y_2)\right) &= \max\left(d_X(x_1, x_2), d_Y(y_1, y_2)\right) \\ &= \max\left(d_X(x_2, x_1), d_Y(y_2, y_1)\right) \\ &= d_{X\times Y}\left((x_2, y_2), (x_1, y_1)\right), \end{aligned}$$

and we have proved **MET 2** for $d_{X\times Y}$.

MET 3. The triangle inequality. If we put in (x,y), we get

$$\begin{aligned} d_X(x_1,x_2) &\leq d_X(x_1,x)+d_X(x,x_2) \\ &\leq d_{X\times Y}\left((x_1,y_1),(x,y)\right)+d_{X\times Y}\left((x,y),(x_2,y_2)\right), \end{aligned}$$

and analogously,

$$d_Y(y_1,y_2)\leq d_{X\times Y}\left((x_1,y_1),(x,y)\right)+d_{X\times Y}\left((x,y),(x_2,y_2)\right).$$

Hence the largest of the numbers

$$d_X(x_1,x_2) \qquad \text{and} \qquad d_Y(y_1,y_2)$$

must be smaller than or equal to the common right hand side, thus

$$\begin{aligned} d_{X\times Y}\left((x_1,y_1),(x_2,y_2)\right) &= \max\left(d_X(x_1,x_2),d_Y(y_1,y_2)\right) \\ &\leq d_{X\times Y}\left((x_1,y_1),(x,y)\right)+d_{X\times Y}\left((x,y),(x_2,y_2)\right), \end{aligned}$$

and **MET 3** is proved.

Summing up, we have proved that $d_{X\times Y}$ is a metric on $X\times Y$.

2. Since $p_X : X\times Y\to X$ fulfils

$$d_X\left(p_X((x,y)),p_X\left((x_0,y_0)\right)\right)=d_X(x,x_0)\leq d_{X\times Y}\left((x,y),(x_0,y_0)\right),$$

we can to every $\varepsilon>0$ choose $\delta=\varepsilon$. Then it follows from $d_{X\times Y}\left((x,y),(x_0,y_0)\right)<\varepsilon$ that

$$d_X\left(p_X((x,y)),p_X\left((x_0,y_0)\right)\right)\leq d_{X\times Y}\left((x,y),(x_0,y_0)\right)<\varepsilon,$$

and we have proved that p_X is continuous.

The proof of $p_Y : X\times Y\to Y$ also being continuous, is analogous.

Example 1.2 *Let (S,d) be a metric space. For every pair of points x, $y\in S$, we set*

$$\overline{d}(x,y)=\frac{d(x,y)}{1+d(x,y)}.$$

Show that $\overline{d}$ is a metric on S with the property

$$0\leq\overline{d}(x,y)<1 \qquad \textit{for all } x,\ y\in S.$$

Hint: *You may in suitable way use that the function $\varphi:\mathbb{R}_0^+\to\mathbb{R}_0^+$ defined by*

$$\varphi(t)=\frac{t}{1+t}, \qquad t\in\mathbb{R}_0^+,$$

is increasing.

MET 1. Obviously,

$$\overline{d}(x,y)=\frac{d(x,y)}{1+d(x,y)}\geq 0,$$

and if $\overline{d}(x,y)=0$, then $d(x,y)=0$, hence $x=y$.

MET 2. From $d(x,y)=d(y,x)$ follows that

$$\overline{d}(x,y)=\frac{d(x,y)}{1+d(x,y)}=\frac{d(y,x)}{1+d(y,x)}=\overline{d}(y,x).$$

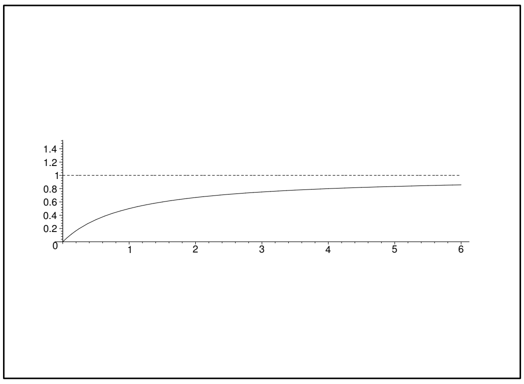


Figure 2: The graph of $\varphi(t)$ and its horizontal asymptote.

MET 3. We shall now turn to the triangle inequality,

$$\overline{d}(x,y) \leq \overline{d}(x,z) + \overline{d}(z,x).$$

Now,

$$d(x,y) \leq d(x,z) + d(z,y),$$

and

$$\varphi(t) = \frac{t}{1+t} = 1 - \frac{1}{1+t} \in [0,1[\qquad \text{for } t \geq 0,$$

is increasing. Since a positive fraction is increased, if its positive denominator is decreased (though still positive), it follows that

$$\begin{aligned}
\overline{d}(x,y) &= \frac{d(x,y)}{1+d(x,y)} = \varphi(d(x,y)) \\
&\leq \varphi(d(x,z)+d(z,x)) = \frac{d(x,z)+d(z,y)}{1+d(x,z)+d(z,y)} \\
&= \frac{d(x,z)}{1+d(x,z)+d(z,y)} + \frac{d(z,y)}{1+d(x,z)+d(z,y)} \\
&= \frac{d(x,z)}{1+d(x,z)} + \frac{d(z,y)}{1+d(z,y)} \\
&= \overline{d}(x,z) + \overline{d}(z,y),
\end{aligned}$$

and we have proved that $\overline{d}$ is a metric.

Now, $\varphi(t) \in [0,1]$ for $t \in \mathbb{R}_0^+$, thus

$$\overline{d}(x,y) = \varphi(d(x,y)) \in [0,1[\quad \text{for all } x,\, y \in S,$$

hence

$$0 \leq \overline{d}(x,y) < 1 \qquad \text{for all } x,\, y \in S.$$

Remark 1.1 Let $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ satisfy the following three conditions:

1. $\varphi(0) = 0$, and $\varphi(t) > 0$ for $t > 0$,
2. φ is increasing
3. $0 \leq \varphi(t+u) \leq \varphi(t) + \varphi(u)$ for all $t,\, u \in \mathbb{R}_0^+$.

If d is a metric on S, then $\varphi \circ d$ is also a metric on S.

The proof which follows the above, is left to the reader. $\Diamond$

Example 1.3 *Let K be an arbitrary set, and let (S,d) be a metric space, in which $0 \le d(x,y) \le 1$ for all x, $y \in S$.*
Let $F(K,S)$ denote the set of mappings $f : K \to S$.
Define $D : F(K,S) \times F(K,S) \to \mathbb{R}_0^+$ by

$$D(f,g) = \sup_{t \in K} d(f(t),g(t)).$$

1. *Show that D is a metric on $F(K,S)$.*
2. *Let $t_0 \in K$ be a fixed point in K and define*

$$Ev_{t_0} : F(K,S) \to S \qquad by \qquad Ev_{t_0}(f) = f(t_0).$$

Show that Ev_{t_0} is continuous.
(Ev_{t_0} is called an evolution *map.)*

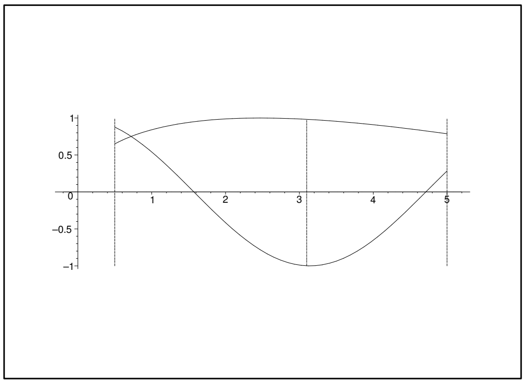


Figure 3: The metric D measures the largest point-wise distance d between the graphs of two functions over each point in the domain $t \in K$.

First notice that since $0 \le d(x,y) \le 1$, we have

$$D(f,g) = \sup_{t \in K} d(f(t),g(t)) \le 1 \qquad \text{for all } f,\ g \in F(K,S).$$

Without a condition of boundedness the supremum could give us $+\infty$, and D would *not* be defined on all of $F(K,S) \times F(K,S)$.

1. **MET 1.** Clearly, $D(f,g) \ge 0$. Assume now that

$$D(f,g) = \sup_{t \in K} d(f(t),g(t)) = 0.$$

Then

$$d(f(t),g(t)) = 0 \qquad \text{for all } t \in K,$$

thus $f(t) = g(t)$ for all $t \in K$. This means that $f = g$, and **MET 1** is proved.

MET 2. is obvious, because

$$D(f,g) = \sup_{t \in K} d(f(t),g(t)) = \sup_{t \in K} d(g(t),f(t)) = D(g,f).$$

MET 3. It follows from

$$d(f(t),g(t)) \le d(f(t),h(t)) + d(h(t),g(t)) \qquad \text{for all } t \in K,$$

that

$$D(f,g) = \sup_{t \in K} d(f(t),g(t)) \le \sup_{t \in K}\{d(f(t),h(t)) + d(h(t),g(t))\}.$$

The maximum/supremum of a sum is of course at most equal to the sum of each of the maxima/suprema, so we continue the estimate by

$$D(f,g) \le \sup_{t\in K} d(f(t),h(t)) + \sup_{t\in K} d(h(t),g(t)) = D(f,h) + D(g,h),$$

and **MET 3** is proved.

Summing up, we have proved that D is a metric on $F(K,S)$.

2. Since

$$d\left(Ev_{t_0}(f), Ev_{t_0}(g)\right) = d\left(f(t_0), g(t_0)\right) \le \sup_{t\in K} d(f(t),g(t)) = D(f,g),$$

we can to every $\varepsilon > 0$ choose $\delta = \varepsilon$, such that if

$$D(f,g) < \delta = \varepsilon,$$

then

$$d\left(Ev_{t_0}(f), Ev_{t_0}(g)\right) \le D(f,g) < \varepsilon,$$

and the map $Ev_{t_0} : F(K,S) \to D$ is continuous.

Example 1.4 *Example 1.1 (2) and Example 1.3 (2) are both special cases of a general result. Try to formulate such a general result.*

Let (X, d_X) and (T, d_Y) be two metric spaces, and let $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a continuous and strictly increasing map (at least in a non-empty interval of the form $[0, a]$) with $\varphi(0) = 0$. Then the inverse map $\varphi^{-1} : [0, \varphi(a)] \to [0, a]$ exists, and is continuous and strictly increasing with $\varphi^{-1}(0) = 0$.

Theorem 1.1 *Let $f : X \to Y$ be a map. If*

$$d_Y(f(x), f(y)) \le \varphi\left(d_X(x, y)\right) \qquad \text{for all } x,\, y \in X,$$

then f is continuous.

Proof. We may without loss of generality assume that $0 < \varepsilon < a$. Choose $\delta = \varphi^{-1}(\varepsilon)$. If x, $y \in X$ satisfy

$$d_X(x, y) < \delta = \varphi^{-1}(\varepsilon),$$

then we have for the image points that

$$d_Y(f(x), f(y)) \le \varphi\left(d_X(x, y)\right) < \varphi\left(\varphi^{-1}(\varepsilon)\right) = \varepsilon,$$

and it follows that f is continuous.

Examples.

1. In the previous two examples, $\varphi(t) = t$, $t \in \mathbb{R}_0^+$. Clearly, φ is continuous and strictly increasing, and $\varphi(0) = 0$.

2. Another example is given by $\varphi(t) = c \cdot t$, $t \in \mathbb{R}_0^+$, where $c > 0$ is a constant.

3. Of more sophisticated examples we choose

$$\varphi(1) = \sqrt{t}, \qquad \varphi(t) = \exp(t) - 1, \qquad \varphi(t) = \ln(t+1),$$

$$\varphi(t) = \sinh(t), \qquad \varphi(t) = \tanh t, \qquad \varphi(t) = \text{ Arctan } t,$$

etc. etc..

2 Topology 1

Example 2.1 *Let (S, d) be a metric space. For $x \in S$ and $r \in \mathbb{R}^+$ let $B_r(x)$ denote the open ball in S with centre x and radius r. Show that the system of open balls in S has the following properties:*

1. *If $y \in B_r(x)$ then $x \in B_r(y)$.*
2. *If $y \in B_r(x)$ and $0 < s \le r - d(x, y)$, then $B_s(y) \subseteqq B_r(x)$.*
3. *If $d(x, y) \ge r + s$, where x, $y \in S$, and r, $s \in \mathbb{R}^+$, then $B_r(x)$ and $B_s(y)$ are mutually disjoint.*

We define as usual

$$B_r(x) = \{y \in S \mid d(x, y) < r\}.$$

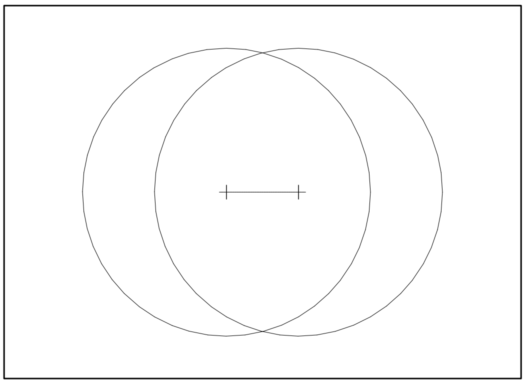


Figure 4: The two balls $B_r(x)$ and $B_r(y)$ and the line between the centres x and y. Notice that this line lies in both balls.

1. If $y \in B_r(x)$, then it follows from the above that $d(x, y) < r$. Then also $d(y, x) < r$, which we interpret as $x \in B_r(y)$.

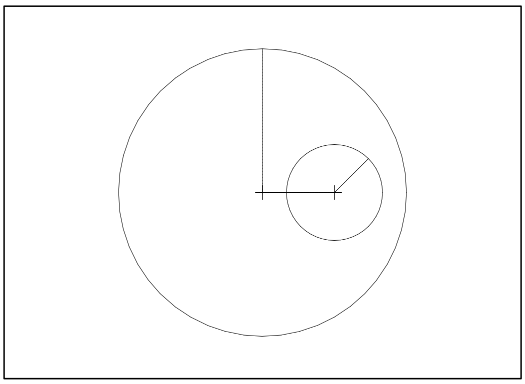


Figure 5: The larger ball $B_r(x)$ contains the smaller ball $B_s(y)$, if only $0 < s \le r - d(x, y)$.

2. If $z \in B_s(y)$, then it follows from the triangle inequality that

$$d(x, z) \le d(x, y) + d(y, z) < d(x, y) + s \le d(x, y) + \{r - d(x, y)\} = r,$$

which shows that $z \in B_r(x)$. This is true for every $z \in B_s(y)$, hence

$$B_s(y) \subseteqq B_r(x).$$

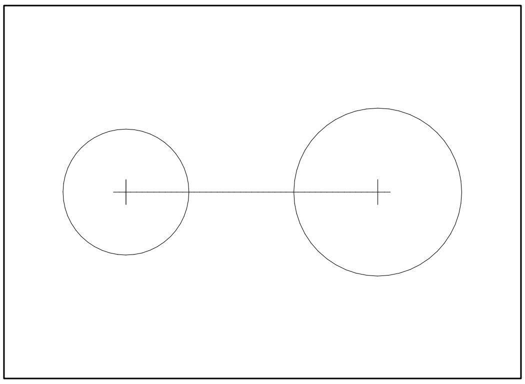


Figure 6: Two balls of radii r and s resp., where $d(x, y) \geq r + s$.

3. *Indirect proof.* Assume that the two balls are *not* disjoint. Then there exists a $z \in B_r(x) \cap B_s(y)$. We infer from the assumption $d(x, y) \geq r + s$ and the triangle inequality that

$$r + s \leq d(x, y) \leq d(x, z) + d(x, y) < r + s,$$

thus $r + r < r + s$, which is a *contradiction*. Hence our *assumption* is false, and we conclude that $B_r(x)$, and $B_s(y)$ are disjoint.

Example 2.2 *Let (S,d) be a metric space. A subset K in S is called* bounded *in (S,d), if there exists a point $x \in S$ and an $r \in \mathbb{R}^+$ such that $K \subseteqq B_r(x)$.*
Examine the truth of each of the following three statements:

1. *If two subsets K_1 and K_2 in S are bounded in (S,d), then their union $K_1 \cup K_2$ is also bounded in (S,d).*

2. *If $K \subseteqq S$ is bounded in (S,d) then*

$$K' = \bigcup_{x \in K} \{y \in S \mid d(x,y) \le 1\}$$

is also bounded in (S,d).

3. *If $K \subseteq S$ is bounded in (S,d) then*

$$K'' = \bigcap_{x \in K} \{y \in S \mid d(x,y) > 1\}$$

is also bounded in (S,d).

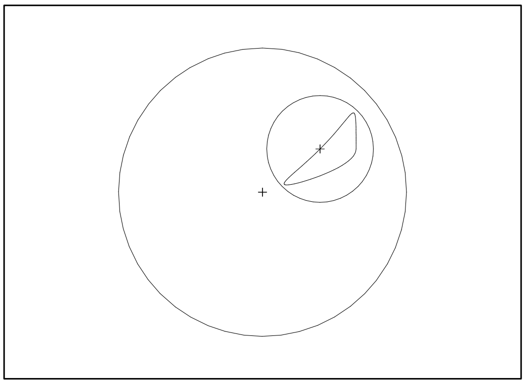


Figure 7: The smaller disc is caught by the larger disc of centre x_0, if only its radius is sufficiently large.

Here there are several possibilities of solution. The elegant solution applies that a set K is bounded, if there exists an $R \in \mathbb{R}^+$, such that $K \subseteqq B_R(x_0)$, where $x_0 \in S$ is a fixed point, which can be used for *every* bounded subset. In fact, if $K \subseteqq B_r(x)$, then $d(y,x) < r$ for all $y \in K$. Then by the triangle inequality

$$d(y,x_0) \le d(y,x) + d(x,x_0) < d(x,x_0) + r = R(x),$$

thus $K \subseteqq B_R(x_0) = B_{R(x)}(x_0)$.

1. **First solution.** If K_1 and K_2 are bounded subsets, then we get with the *same* reference point $x_0 \in S$,

$$K_1 \subseteqq B_{R_1}(x_0) \qquad \text{and} \qquad K_2 \subseteqq B_{R_2}(x_0),$$

hence

$$K_1 \cup K_2 \subseteqq B_{R_1}(x_0) \cup B_{R_2}(x_0) = B_{\max\{R_1,R_2\}}(x_0) = B_R(x_0).$$

Now $R = \max\{R_1, R_2\} < +\infty$, so it follows that the union $K_1 \cup K_2$ is bounded, when both K_1 and K_2 are bounded.

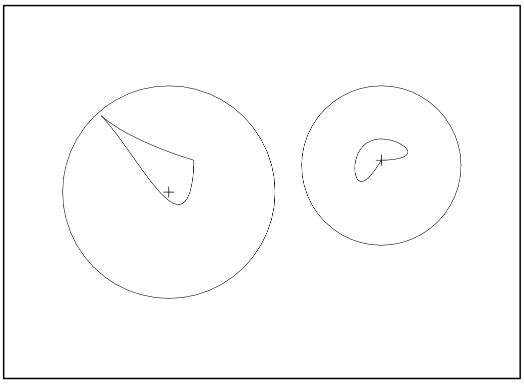


Figure 8: A graphic description of the second solution.

Second solution. Here we give a proof which is closer to the definition. First note that there are x, $y \in S$ and r, $s > 0$, such that

$$K_1 \subseteqq B_r(x) \qquad \text{and} \qquad K_2 \subseteqq B_s(y).$$

Choosing $R = r + d(x,y) + s > r$, it is obvious that since the radius is increased and the centre is the same

$$K_1 \subseteqq B_r(x) \subseteqq B_R(x).$$

Then apply a result from Example 2.1 (2),

$$K_2 \subseteqq B_s(y) \subseteqq B_{r+d(x,y)+s}(x) = B_R(x),$$

and we see that $K_1 \cup K_2 \subseteqq B_R(x) \cup B_R(x) = B_R(x)$ is bounded.

Alternatively, it follows for every $z \in B_s(y)$ that

$$d(x,z) \leq d(x,y) + d(y,z) < d(x,y) + s < R,$$

så $K_2 \subseteqq B_s(y) \subseteqq B_R(x)$.

2. Now K is bounded, so $K \subseteqq B_R(x_0)$, and

$$K' \subseteqq B_{R+1}(x_0).$$

In fact, if $y \in K'$, then we can find an $x \in K$, such that $d(x,y) \leq 1$. Since $x \in K \subseteqq B_R(x_0)$, we have $d(x, x_0) < R$. Thus

$$d(y,x_0) \leq d(y,x) + d(x,x_0) < R+1,$$

and therefore $y \in B_{R+1}(x_0)$. This holds for every $y \in K'$, so $K' \subseteqq B_{R+1}(x_0)$, and K' is bounded.

3. **First possibility; the metric d is bounded.** In this case there is a constant $c > 0$, such that

$$d(x,y) \leq c < +\infty \qquad \text{for all } x,\ y \in S.$$

In particular, S is itself bounded,

$$S = B_c(x), \qquad \text{for every } x \in S.$$

Every subset of S is bounded.

Second possibility; the metric d is unbounded. In this case the claim is not true. In fact, the complementary set of K''

$$(1) \qquad S \setminus K'' = \bigcup_{x \in K} \{y \in S \mid d(x,y) \leq 1\} = K'$$

is bounded according to the second question. Then $S = K' \cup K''$ is a disjoint union, and since K' is bounded, while S is unbounded, we conclude that K'' is also unbounded. (Otherwise $K' \cup K''$ would be bounded by the first question).

Remark 2.1 Proof of (1). If

$$y \notin K'' = \bigcap_{x \in K} \{y \in S \mid d(x,y) > 1\},$$

then there exists an $x \in K$, such that $d(x,y) \le 1$, and $\bigcap$ is replaced by $\bigcup$, and $d(x,y) > 1$ is replaced by the negation $d(x,y) \le 1$, and (1) follows. $\Diamond$

Example 2.3 *List all topologies that can be defined on a set $S = \{a, b\}$ containing only two elements a and b.*

Every topology must contain at least the empty set $\emptyset$ and the total space $S = \{a, b\}$. The only *possibilities* are

$$\mathcal{T}_1 = \{\emptyset, S\}, \qquad \mathcal{T}_2 = \{\emptyset, \{a\}, S\}, \qquad \mathcal{T}_3 = \{\emptyset, \{b\}, S\},$$

$$\mathcal{T}_4 = \mathcal{D}(S) = \{\emptyset, \{a\}, \{b\}, S\},$$

where $\mathcal{D}(S)$ denotes the set of all subsets of S. It is well-known that $\mathcal{T}_1$ and $\mathcal{T}_4$ are topologies, called the coarsest and the finest topology on S).

Since any union and even any intersection of sets from $\mathcal{T}_2$ again belong to $\mathcal{T}_2$, it follows that $\mathcal{T}_2$ is a topology.

Analogously for $\mathcal{T}_3$ (exchange a by b).

The four possibilities above are therefore all possible topologies on $S = \{a, b\}$.

Example 2.4 *Let $\mathcal{T}$ be the system of subsets U in $\mathbb{R}$ which is one of the following types:*
Either

(i) *U does not contain 0,*

or

(ii) *U does contain 0, and the complementary set $\mathbb{R} \setminus U$ is finite.*

1. *Show that $\mathcal{T}$ is a topology on $\mathbb{R}$.*
2. *Show that $\mathbb{R}$ with the topology $\mathcal{T}$ is a Hausdorff space.*

   (A topological space $(S, \mathcal{T})$ is called a Hausdorff space, if one to any pair of points x, $y \in S$, where $x \neq y$, can find a corresponding pair of disjoint open sets U, $V \in \mathcal{T}$, such that $x \in U$ and $y \in V$)
3. *Prove that the topology $\mathcal{T}$ on $\mathbb{R}$ is not generated by a metric on $\mathbb{R}$, because there does not exist any countable system of open neighbourhoods of $0 \in \mathbb{R}$ in the topology $\mathcal{T}$ with the property that any arbitrary open set of $0 \in \mathbb{R}$ contains a neighbourhood from this system.*

1. We shall prove that

   TOP 1. If $\{U_i \in \mathcal{T} \mid i \in\} \subset \mathcal{T}$, then $\bigcup_{i \in I} U_i \in \mathcal{T}$.

   TOP 2. If $U_i \in \mathcal{T}$, $i = 1, \dots, k$, then $\bigcap_{i=1}^{k} U_i \in \mathcal{T}$.

   TOP 1. $\emptyset, \mathbb{R} \in \mathcal{T}$.

   We go through them one by one.

   TOP 1. Let $\{U_i \in \mathcal{T} \mid i \in I\}$ be any family of sets from $\mathcal{T}$.

   (i) If no U_i, $i \in I$, contains 0, then $0 \notin \bigcup_{i \in I} U_i$, which means that $\bigcup_{i \in I} U_i \in \mathcal{T}$.

   (ii) If (at least) one U_i contains 0, and $\mathbb{R} \setminus U_i$ is finite, then

   $$0 \in \bigcup_{i \in I} U_i \quad \text{and} \quad \mathbb{R} \setminus \bigcup_{i \in I} U_i \subseteqq \mathbb{R} \setminus U_i \text{ is finite.}$$

   This proves that $\bigcup_{i \in I} U_i \in \mathcal{T}$.

   Summing up, we have proved condition **TOP 1** for a topology.

   TOP 2. Let $\{U_i \in \mathcal{T} \mid i = 1, \dots, k\}$ be a *finite* family of sets from $\mathcal{T}$. We shall start by considering a system of sets, which all satisfy (ii).

(ii) If $0 \in U_i$, and $\mathbb{R} \setminus U_i$ is finite for every $i = 1, \ldots, k$, then $0 \in \bigcap_{i=1}^{k} U_k$, and

$$\mathbb{R} \setminus \bigcap_{i=1}^{k} U_i = \bigcup_{i=1}^{k} \mathbb{R} \setminus U_i$$

is a finite union of finite sets, hence itself finite.

Alternatively, the *longer* version is the following: If $\mathbb{R} \setminus U_i$ contains n_i different elements, then $\bigcup_{i=1}^{k} \mathbb{R} \setminus U_i$ contains at most $n = \sum_{i=1}^{k} n_i < +\infty$ different elements.

In this case we conclude that $\bigcap_{i=1}^{k} U_i \in \mathcal{T}$.

(i) If there is an U_i, where $i \in \{, \ldots, k\}$, such that $0 \notin U_i$, (notice that we are not at all concerned with the other sets U_j being open of type (i) or type (ii); we shall just have one open set of type (i)), then clearly $0 \notin \bigcap_{i=1}^{k} U_i$, thus $\bigcap_{i=1}^{k} U_i \in \mathcal{T}$.

Summing up we have proved condition **TOP 2** for a topology.

TOP 3. From $0 \notin \emptyset$ follows from (i) that $\emptyset \in \mathcal{T}$. From $0 \in \mathbb{R}$ and $\mathbb{R} \setminus \mathbb{R} = \emptyset$ containing no element it follows by (ii) that $\mathbb{R} \in \mathcal{T}$, and we have proved the remaining condition **TOP 3** for a topology.

Summing up we have proved that $\mathcal{T}$ is a topology.

2. We shall now prove that the space $(\mathbb{R}, \mathcal{T})$ is a Hausdorff space.

   (i) If x, $y \in \mathbb{R} \setminus \{0\}$, then $\{x\}$, $\{y\} \in \mathcal{T}$ by definition (i). Furthermore, if $x \neq y$, then clearly $\{x\} \cap \{y\} = \emptyset$.

   (ii) If $x \in \mathbb{R} \setminus \{0\}$ and $y = 0$, then $\{x\}$, $\mathbb{R} \setminus \{x\} \in \mathcal{T}$ by (i) and (ii), resp., and $0 \in \mathbb{R} \setminus \{x\}$, and $\{x\} \cap (\mathbb{R} \setminus \{x\}) = \emptyset$.

   We have proved that the space is a Hausdorff space.

3. Assume that $\{U_n \mid n \in \mathbb{N}\}$ is a countable system of open neighbourhoods of 0, thus $0 \in U_n$, and $\mathbb{R} \setminus U_n$ is finite. Then the "exceptional set"

$$A = \bigcup_{n=1}^{\infty} \mathbb{R} \setminus U_n$$

   is at most countable. In particular, $A \neq \mathbb{R}$.

   Choose any point $a \in \mathbb{R} \setminus (A \cup \{0\})$. Then $U = \mathbb{R} \setminus \{a\} \in \mathcal{T}$ is a neighbourhood of 0, and none of the U_n is contained in U.

   In fact, if $U_n \subseteq U$, then

$$\{a\} = \mathbb{R} \setminus U \subseteq \mathbb{R} \setminus U_n \subseteq \bigcup_{n=1}^{\infty} \mathbb{R} \setminus U_n = A,$$

   which is a contradiction.

3 Continuous mappings

Example 3.1 *Let S be a topological space with topology $\mathcal{T}$, and let $\pi : S \to \tilde{S}$ be a mapping into a set $\tilde{S}$. Let $\tilde{\mathcal{T}}$ be the quotient topology induced from the topology $\mathcal{T}$ on S by the mapping π.*

1. *Let $\mathcal{T}'$ be a topology on $\tilde{S}$, such that $\pi : S \to \tilde{S}$ is continuous when S is considered with the topology $\mathcal{T}$ and $\tilde{S}$ with the topology $\mathcal{T}'$.*

   Show that $\mathcal{T}' \subseteqq \tilde{\mathcal{T}}$.

   (The quotient topology $\tilde{\mathcal{T}}$ on $\tilde{S}$ is in other words the 'largest' topology on $\tilde{S}$ for which $\pi : S \to \tilde{S}$ is continuous.)

2. *Show that when $\tilde{S}$ has the quotient topology determined by the mapping $\pi : S \to \tilde{S}$, then the following holds:*

   - *A mapping $f : \tilde{S} \to T$ into a topological space T is continuous if and only if the composite mapping $f \circ \pi : S \to T$ is continuous.*

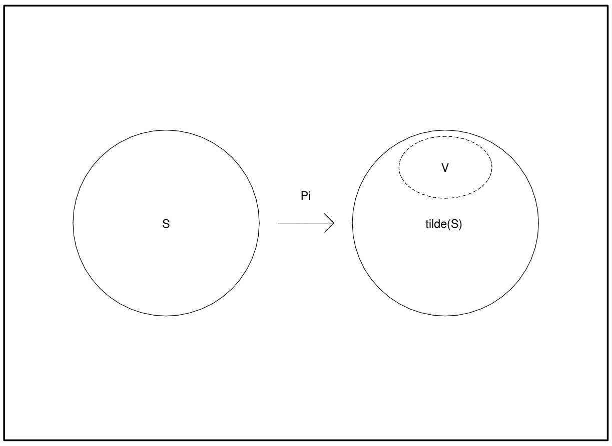


Figure 9: The topology $\mathcal{T}$ is defined on S, and the quotient topology $\tilde{\mathcal{T}}$, or $\mathcal{T}'$, is defined on $\tilde{S}$.

We recall that the quotient topology is defined by

$$\tilde{\mathcal{T}} = \{V \subseteqq \tilde{S} \mid U = \pi^{-1}(V) \in \mathcal{T}\}.$$

1. If $\pi : S \to \tilde{S}$ is continuous in the topology $\mathcal{T}'$ on $\tilde{S}$ and $\mathcal{T}$ on S, then

   $$\pi^{-1}(V) \in \mathcal{T} \qquad \text{for ehvery } V \in \mathcal{T}',$$

   hence $V \in \tilde{\mathcal{T}}$ for every $V \in \mathcal{T}'$. This means precisely that

   $$\mathcal{T}' \subseteqq \tilde{\mathcal{T}}.$$

2. Assume that $f : \tilde{S} \to T$ is continuous, where $\tilde{S}$ has the quotient topology $\tilde{\mathcal{T}}$ determined by $\pi : S \to \tilde{S}$, and where T has the topology $\mathcal{T}^\star$. This means that

   $$f^{-1}(V) \in \tilde{\mathcal{T}} = \{W \subseteqq \tilde{S} \mid \pi^{-1}(W) \in \mathcal{T}\}$$

   for every $V \in \mathcal{T}^\star$.

   Then it follows that

   $$\pi^{-1}(W) = \pi^{-1}\left(f^{-1}(V)\right) = (f \circ \pi)^{-1}(V) \in \mathcal{T}$$

   for every $V \in \mathcal{T}^\star$. This is precisely the condition that the composite mapping $f \circ \pi : S \to T$ is continuous.

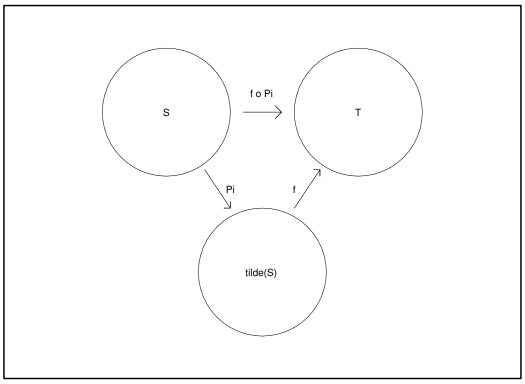


Figure 10: Diagram, where S has the topology $\mathcal{T}$, and $\tilde{S}$ has the topology $\tilde{\mathcal{T}}$, and T has the topology $\cap\mathcal{T}^\star$.

Conversely, if $f \circ \pi : S \to T$ is continuous, then

$$\mathcal{T} \ni (f \circ \pi)^{-1}(V) = \pi^{-1}\left(f^{-1}(V)\right) \qquad \text{for every } V \in \mathcal{T}^\star.$$

Then it follows from the definition of the quotient topology that if $\pi^{-1}\left(f^{-1}(V)\right) \in \mathcal{V}$, then $f^{-1}(V) \in \tilde{\mathcal{T}}$.

Since $f^{-1}(V) \in \tilde{\mathcal{T}}$ for every $V \in \mathcal{T}^\star$, it follows that $f : \tilde{S} \to T$ is continuous, and the claim is proved.

Example 3.2 *Let S be a topological space. For every pair of real-valued functions f, $g : S \to \mathbb{R}$, we can in the usual way define the functions $f+g$, $f-g$, $f \cdot g$, and (if $g(x) \neq 0$ for all $x \in S$) f/g.*

1. *Show that if f and g are continuous at a point $x_0 \in S$, then also $f+g$, $f-g$, $f \cdot g$, and (when it is defined) f/g are continuous at $x_0 \in S$.*

   (Carry through the argument in at least one case.)

2. *Assume that f, $g : S \to \mathbb{R}$ are continuous. Show that*

   $$U = \{x \in S \mid f(x) < g(x)\}$$

   is an open set in S.

3. *Let $f_1, \dots, f_k : S \to \mathbb{R}$ be continuous real-valued functions. Show that*

   $$U = \{x \in S \mid f_i(x) < a_i,\ i = 1, \dots, k\}$$

   is an open set in S, where $a_1, \dots, a_k \in \mathbb{R}$ are real numbers.

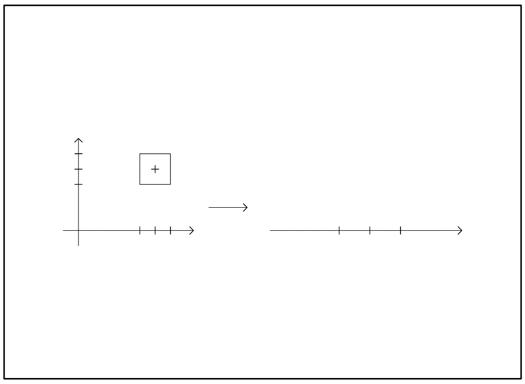


Figure 11: The interval $\left[a+\frac{\varepsilon}{2}, b+\frac{\varepsilon}{2}\right]$ to the left is by addition $+$ mapped into the interval $[a+b-\varepsilon, a+b+\varepsilon]$.

1. In reality, this example is concerned with the continuity of the basic four arithmetical operations

   $$+,\ -,\ \cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \quad \text{og} \quad / : \mathbb{R} \times (\mathbb{R} \setminus \{0\}) \to \mathbb{R}.$$

   The remaining part follows easily by composition of continuous mappings.

   (a) *Addition* $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous.
   Let $(a,b) \in \mathbb{R} \times \mathbb{R}$ be given. To every $\varepsilon > 0$ choose $\delta = \frac{\varepsilon}{2}$. If

   $$|x-a| < \frac{\varepsilon}{2} \qquad \text{and} \qquad |y-b| < \frac{\varepsilon}{2},$$

   then

   $$|(x+y)-(a+b)| \leq |x-a| + |y-b| < \varepsilon,$$

   which is precisely the classical proof of continuity.

   (b) *Subtraction* $- : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ follows the same pattern: If

   $$|x-a| < \frac{\varepsilon}{2} \qquad \text{and} \qquad |y-b| < \frac{\varepsilon}{2},$$

   then

   $$|(x-y)-(a-b)| \leq |x-a| + |y-b| < \varepsilon.$$

(c) *Multiplication* $\cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous.
We first assume that $|x-a| < \delta$ and $|y-b| < \delta$ in order to derive the right relation between δ and ε. From $f(x,y) = x \cdot y$, we get by the triangle inequality by inserting $-ay + av = 0$ that

$$\begin{aligned} |f(x,y) - f(a,b)| &= |xy - an| = |xy - ay + au - ab| \\ &\leq |x-a| \cdot |y| + |a| \cdot |y-b| \leq \delta \cdot |y| + |a| \cdot \delta \\ &\leq \delta(|b| + \delta + |a|). \end{aligned}$$

Since $\varphi(\delta) = \delta(|a| + |b| + \delta)$ is continuous and strictly increasing for $\delta \in \mathbb{R}_0^+$ of the value $\varphi(0) = 0$, the mapping φ has a (continuous) inverse φ^{-1}. By choosing $\delta = \varphi^{-1}(\varepsilon)$, we get precisely

$$|f(x,y) - f(a,b)| \leq \varphi(\delta) = \varepsilon,$$

and the multiplication is continuous.

(d) The mapping $y \to \dfrac{1}{y}$ is continuous for $y \in \mathbb{R} \setminus \{0\}$.
Let $b \neq 0$, and choose y and $\delta \in]0, |b|[$, such that

$$|y - b| < \delta < |b|.$$

Then

$$\left| \frac{1}{y} - \frac{1}{b} \right| = \frac{|y-b|}{|y| \cdot |b|} \leq \frac{\delta}{|b| \cdot (|b| - \delta)}, \qquad \text{for } \delta \in]0, |b|[.$$

It is obvious that to any $\varepsilon > 0$ there is a $\delta > 0$, such that

$$\left| \frac{1}{y} - \frac{1}{b} \right| \leq \frac{\delta}{|b| \cdot (|b| - \delta)} < \varepsilon,$$

and the mapping is continuous.

(e) If the denominator is $\neq 0$, then the *division* is continuous.
This is obvious, because division is composed of the continuous mappings

$$d)\ \ (x,y) \curvearrowright \left(x, \frac{1}{y} \right), \quad \text{and} \quad c)\ \ \left(x, \frac{1}{y} \right) \curvearrowright x \cdot \frac{1}{y} = \frac{x}{y},$$

thus it is itself continuous.

Summing up we have proved that if $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is one of the four basic arithmetical operations $+$, $-$, $\cdot$, $/$ (provided the denominator is $\neq 0$), then φ is continuous. Then the mapping

$$S \xrightarrow{\Delta} S \times S \xrightarrow{f \times g} \mathbb{R} \times \mathbb{R} \xrightarrow{\varphi} \mathbb{R},$$

is also continuous, because the diagonal mapping $\Delta(x) = (x,x)$ is trivially continuous, and because $f \times f$ is continuous at $(x_0, x_0) \in S \times S$.

We have now proved 1).

2. If f, $g : S \to \mathbb{R}$ are both continuous, then $f - g : S \to \mathbb{R}$ is also continuous according to 1). Then

$$U = \{x \in S \mid f(x) < g(x)\} = \{x \in S \mid (f-g)(x) < 0\} = (f-g)^{\circ -1}(] - \infty, 0[)$$

is open, because $\mathbb{R}^- =] - \infty, 0[)$ is open.

3. Each

$$U_i = \{x \in S \mid f_i(x) < a_i\} = f_i^{\circ -1}(] - \infty, a_i[)$$

is open, hence

$$U = \{x \in S \mid f_i(x) < a_i,\ i = 1, \dots, k\} = \bigcap_{i=1}^{k} \{x \in S \mid f_i(x) < a_i\} = \bigcap_{i=1}^{k} U_i$$

is also open as a *finite* intersection of open sets.

Example 3.3 *Let S be a topological space with topology $\mathcal{T}$, and let A be an arbitrary subset in S. Equip A with the induced topology $\mathcal{T}_A$.*
Show that a subset $B' \subseteqq A$ is closed in A with the topology $\mathcal{T}_A$ if and only if there exists a closed subset $B \subseteqq S$ in the topology $\mathcal{T}$ such that $B' = A \cap B$.

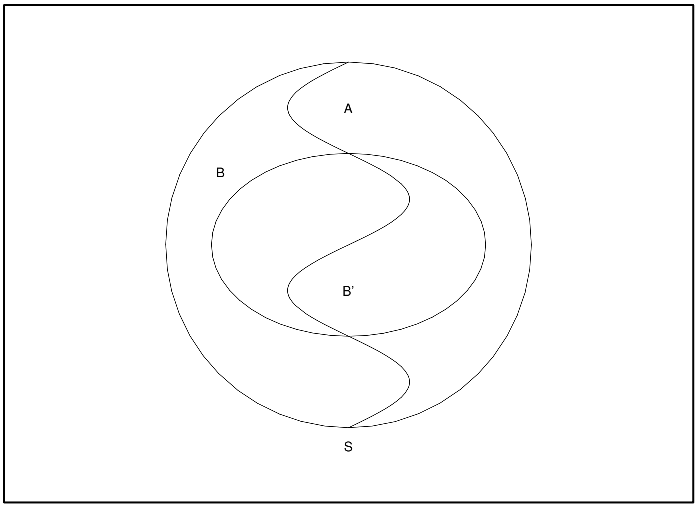


Figure 12: Diagram of the sets of Example 3.3.

The induced topology $\mathcal{T}_A$ is defined by

$$\mathcal{T}_A = \{U \cap A \mid U \in \mathcal{T}\}.$$

1. Assume that $B' \subseteqq A$ is closed in $\mathcal{T}_A$, thus $A \setminus B'$ is open in $\mathcal{T}_A$. By the above there is an $U \in \mathcal{T}$, such that

$$U \cap A = A \setminus B'.$$

Then $B = S \setminus U$ is closed, and

$$B \cap A = A \cap (S \setminus U) = A \setminus U = A \setminus (A \setminus B') = B'.$$

2. Assume conversely that $B' = A \cap B$, where B is closed in S, thus $U = S \setminus B \in \mathcal{T}$ is open. Then

$$U \cap A = A \cap (S \setminus B) = A \setminus B \in \mathcal{T}_A$$

is open in A, hence

$$A \setminus (U \cap A) = A \setminus (A \setminus B) = A \cap B = B'$$

is closed in A, i.e. in the topology $\mathcal{T}_A$.

Example 3.4 *Let $f : X \to Y$ be a mapping between topological spaces X and Y.*
If $f : X \to Y$ maps a subset $X' \subseteqq X$ in X into a subset $Y' \subseteqq Y$ in Y, then f determines a mapping $f' : X' \to Y'$ defined by $f'(x) = f(x)$ for $x \in X'$.
When a subset of a topological space is considered as a topological space in the following, it is always with the induced topology.

1. *Let $f' : X' \to Y'$ be a mapping determined by $f : X \to Y$ as above. Show that if f is continuous, then f' is continuous.*

2. *Let A_1 and A_2 be closed subsets in X such that $X = A_1 \cup A_2$. Let $f_1 : A_1 \to Y$ and $f_2 : A_2 \to Y$ be the mapping determined by f, i.e. the restrictions of f to A_1 and A_2 respectively.*

   Show that if f_1 and f_2 are continuous, then f is continuous.

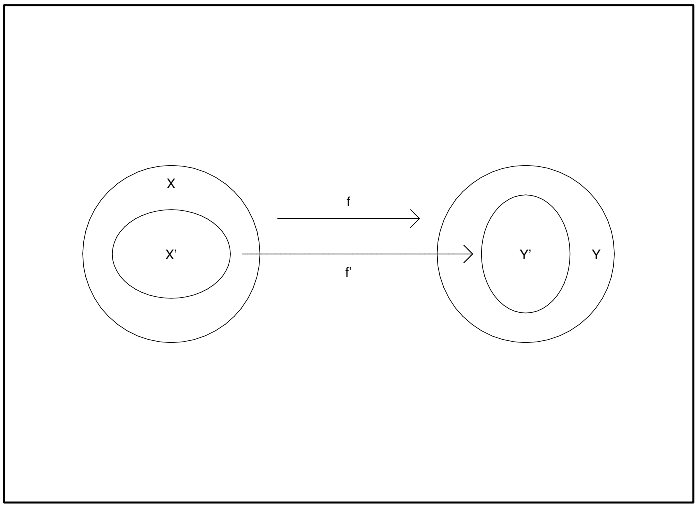


Figure 13: The restriction mapping of the first question.

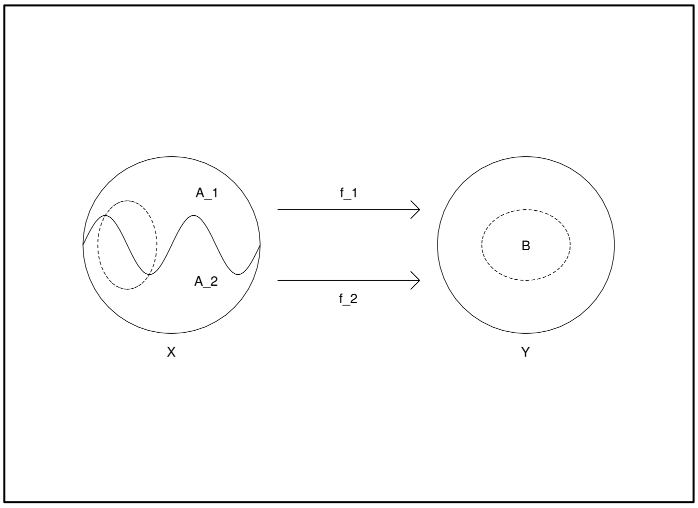


Figure 14: Diagram corresponding to the second question.

1. Let $U' \in \mathcal{T}_{Y'}$ be open in Y', thus there is an $U \in \mathcal{T}_Y$, such that

   $U' = U \cap Y'$.

Since f is continuous and $U \in \mathcal{T}_Y$, we have $f^{\circ -1}(U) \in \mathcal{T}_X$, hence

$$(f')^{\circ -1}(U') = (f')^{\circ -1}(U \cap Y') = f^{\circ -1}(U) \cap X' \in \mathcal{T}_{X'},$$

proving that f' is continuous.

2. Choosing $B \subseteqq Y$ *closed*, we get

$$f^{\circ -1}(B) = f_1^{\circ -1}(B) \cup f_2^{\circ -1}(B),$$

where $f_1^{\circ -1}(B)$ is closed in A_1, and $f_2^{\circ -1}(B)$ is closed in A_2.

Since both A_1 and A_2 are *closed*, it follows that $f_1^{\circ -1}(B)$ and $f_2^{\circ -1}(B)$ are closed in S, thus $f^{\circ -1}(B)$ is closed in S, and f is continuous, where

$$f(x) = \begin{cases} f_1(x), & x \in A_1, \\ f_2(x), & x \in A_2, \end{cases} \qquad A_1,\ A_2 \text{ are closed and disjoint.}$$

4 Topology 2

Example 4.1 *Let W_1 and W_2 be arbitrary subsets in the topological space S. Show that*

1. $\operatorname{int}(W_1 \cap W_2) = \operatorname{int} W_1 \cap \operatorname{int} W_2$.
2. $\operatorname{int}(W_1 \cup W_2) \supseteq \operatorname{int} W_1 \cup \operatorname{int} W_2$.

Give an example that the equality sign in (2) does not apply in general.

1. If $x \in \operatorname{int}(W_1 \cap W_2)$, then there is an open set U in S, such that
$$x \in U \subseteq W_1 \cap W_2,$$
and since $W_1 \cap W_2 \subseteq W_i$, $i = 1, 2$, we get in particular that
$$x \in U \subseteq W_1 \quad \text{and} \quad x \in U \subseteq W_2, \qquad \text{so } x \in \operatorname{int} W_1 \cap \operatorname{int} W_2.$$
This shows that
$$\operatorname{int}(W_1 \cap W_2) \subseteq \operatorname{int} W_2 \cap \operatorname{int} W_2.$$
Conversely, if $x \in \operatorname{int} W_1 \cap \operatorname{int} W_2$, then there exist open sets U_1 and U_2, such that
$$x \in U_1 \subseteq W_1 \qquad \text{and} \qquad x \in U_2 \subseteq W_2.$$
Then $U = U_1 \cap U_2$ is open, and
$$x \in U = U_1 \cap U_2 \subseteq W_1 \cap W_2, \qquad \text{thus } x \in \operatorname{int}(W_1 \cap W_2).$$
It follows that $\operatorname{int}(W_1 \cap W_2) \supseteq \operatorname{int} W_1 \cap \operatorname{int} W_2$, and we have proved that
$$\operatorname{int}(W_1 \cap W_2) = \operatorname{int} W_1 \cap \operatorname{int} W_2.$$
2. We get from $W_1 \subseteq W_1 \cup W_2$ and $W_2 \subseteq W_1 \cup W_2$ that
$$\operatorname{int} W_1 \subseteq \operatorname{int}(W_1 \cup W_2) \quad \text{og} \quad \operatorname{int} W_2 \subseteq \operatorname{int}(W_1 \cup W_2),$$
hence by taking the union,
$$\operatorname{int} W_1 \cup \operatorname{int} W_2 \subseteq \operatorname{int}(W_1 \cup W_2).$$
3. We do not always have equality here. An extreme example is
$$W_1 = \mathbb{Q} \subset \mathbb{R} \qquad \text{and} \qquad W_2 = \mathbb{R} \setminus \mathbb{Q} \subset \mathbb{R},$$
i.e. the rational numbers and the irrational numbers. Then $\operatorname{int} \mathbb{Q} = \emptyset$ and $\operatorname{int}(\mathbb{R} \setminus \mathbb{Q}) = \emptyset$, hence
$$\emptyset = \operatorname{int}(W_1) \cup \operatorname{int}(W_2) \subset \operatorname{int}(\mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q})) = \operatorname{int}(\mathbb{R}) = \mathbb{R},$$
and we do not have equality.

Example 4.2 *Show that a topological space S is a T_1-space if and only if every subset in S containing exactly one point is a closed subset.*

Recall that S is a T_1-rum, if to any pair $x, y \in S$ of different points, $x \neq y$, there exists an open neighbourhood V of y, such that $x \notin V$.

1. Assume that all singletons $\{x\}$, $x \in S$, are closed. Then $S \setminus \{x\}$ is open.

   If $x, y \in S$ and $x \neq y$, choose $U = S \setminus \{y\}$ as an open neighbourhood of x, and $V = S \setminus \{x\}$ as an open neighbourhood of y. Then clearly, $y \notin U$ and $x \notin V$, and S is a T_1-space.

2. Conversely, assume that e.g. $\{x\}$ is *not* closed. Then the closure $\overline{\{x\}}$ contains a point $y \in \overline{\{x\}} \setminus \{x\} \neq \emptyset$, and $\overline{\{x\}}$ is the smallest closed set which contains x.

   If S were a T_1-rum, then there would be an open neighbourhood V of y, which does not contain x. Then $\overline{\{x\}} \cap (S \setminus V)$ would be closed (as an intersection of two closed sets), non-empty (because x lies in both sets), and certainly contained in $\overline{\{x\}}$, i.e.

$$\emptyset \neq \overline{\{x\}} \cap (S \setminus V) \subsetneqq \overline{\{x\}} \setminus \{y\} \begin{cases} \subset \overline{\{x\}}, \\ \neq \overline{\{x\}}. \end{cases}$$

   This is not possible because $\overline{\{x\}}$ is defined as the smallest closed set containing $\{x\}$.

   Hence, if S is a T_1-space, then every point $\{x\}$ is closed.

Example 4.3 *Let S be a Hausdorff space, and let W be an arbitrary subset of S. (It is sufficient that S satisfies the separation property T_1).*
Prove that if $x \in S$ is an accumulation point of W, then every neighbourhod of x in S contains infinitely many different points of W.

An accumulation point $x \in S$ of W is a point for which every neighbourhood U of x (in S) contains at least one point $y \in W$, where $y \neq x$.

Let U_1 be any *open* neighbourhood of x, and choose $y_1 \in W \cap U_1$, such that $y_1 \neq x$. It follows from EXAMPLE 4.2 that $\{y_1\}$ is closed, if S is just a T_1-space. Then $U_2 = U_1 \setminus \{y_1\}$ is an *open* neighbourhood of x, and we can choose $y_2 \in W \cap U_2 \setminus \{x\}$, i.e. $y_2 \neq x$ and $y_2 \neq y_1$.

Then consider the open set $U_3 = U_1 \setminus \{y_1, y_2\}$, etc.

In the n-th step we have an open neighbourhood

$$U_n = U_{n-1} \setminus \{y_1, y_2, \dots, y_{n-1}\}$$

of x, where y_1, y_2, $\dots$, $y_{n-1} \in W$ are mutually different, and where each of them is different from x. Then choose $y_n \in U_n \cap W$, such that $y_n \neq x$, and y_n different from all the previous chosen elements $\{y_1, y_2, \dots, y_2\}$.

Since

$$U_{n+1} = U_1 \setminus \{y_1, y_2, \dots, y_n\} \text{ is open and } x \in U_{n+1} \neq \emptyset,$$

the process never stops, and we have proved that any open neighbourhood of x contains infinitely many different elements from W.

If U is any neighbourhood of x, then it contains an *open* neighbourhood U_1 of x, thus $x \in U_1 \subseteqq U$. Since already U_1 has the wanted property, the larger set U will also have it.

Example 4.4 *Let (S, d) be a metric space. For an arbitrary non-empty subset W in S, we define a function $\varphi : S \to \mathbb{R}$ by*

$$\varphi(x) = \inf\{d(x,y) \mid y \in W\} \qquad \text{for } x \in S.$$

We call $\varphi(x)$ the distance *from x to W, and write, accordingly,*

$$\varphi(x) = d(x, W).$$

1. *Let x_1, $x_2 \in S$ be arbitrary points in S.*
   First show that for an arbitrary point $y \in W$ it holds that

   $$\varphi(x_1) \le d(x_1, x_2) + d(x_2, y).$$

   Next show that

   $$|\varphi(x_1) - \varphi(x_2)| \le d(x_1, x_2),$$

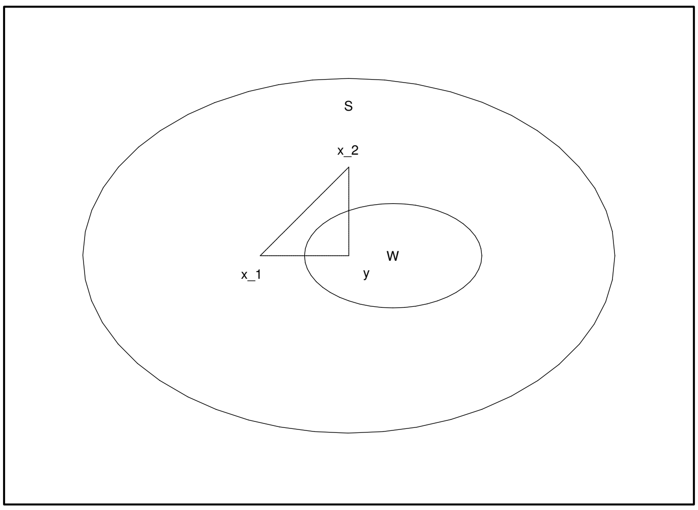
   and conclude from this fact that φ is (uniformly) continuous on S.

2. *Show that*

   $$d(x, W) = \varphi(x) \iff x \in \overline{W},$$

   where $\overline{W}$ as usual denotes the closure of W.

3. *Let A_1 and A_2 be disjoint, non-empty closed subsets in the metric space (S, d). Show that there exist disjoint, open sets U_1 and U_2 in S, such that $A_1 \subseteqq U_1$ and $A_2 \subseteqq U_2$.*
   Hint: Consider the distance functions $\varphi_1(x) = d(x, A_1)$ and $\varphi_2(x) = d(x, A_2)$.



1. This example is an exercise in the triangle inequality. Let $y \in W$ and x_1, $x_2 \in S$. Then

$$\varphi(x_1) = \inf\{d(x_1, \tilde{y}) \mid \tilde{y} \in W\} \leq d(x_1, y) \leq d(x_1, x_2) + d(x_2, y).$$

It follows from

$$\varphi(x_1) \leq d(x_1, x_2) + d(x_2, y) \qquad \text{for every } y \in W,$$

that

$$\varphi(x_1) \leq d(x_1, x_2) + \inf\{d(x_2, y) \mid y \in W\} = d(x_1, x_2) + \varphi(x_2),$$

hence

$$\varphi(x_1) - \varphi(x_2) \leq d(x_1, x_2).$$

An interchange of x_1 and x_2 gives

$$\varphi(x_2) - \varphi(x_1) \leq d(x_2, x_1) = d(x_1, x_2),$$

hence

$$|\varphi(x_1) - \varphi(x_2)| \leq d(x_1, x_2).$$

Now we can independently of the points x_1 and $x_2 \in S$ to every $\varepsilon > 0$ choose $\delta = \varepsilon > 0$, such that

$$d(x_1, x_2) < \varepsilon \quad \text{implies that} \quad |\varphi(x_1) - \varphi(x_2)| < \varepsilon,$$

hence φ is uniformly continuous.

2. Assume that $\varphi(x) = 0$, i.e.

$$\varphi(x) = \inf\{d(x, y) \mid y \in W\} = 0.$$

Then there exists a sequence $\{y_n\} \subseteq W$, such that $d(x, y_n) < \dfrac{1}{n}$, and every open ball $B_{1/n}(x)$ of centre x and radius $\dfrac{1}{n}$ contains points from W,

$$W \cap B_{1/n}(x) \neq \emptyset \qquad \text{for every } n \in \mathbb{N}.$$

Then $x \in \overline{W}$.

Conversely, if $x \in \overline{W}$, then there exists a sequence $\{y_n\} \subseteq W$, such that

$$d(x, y_n) < \frac{1}{n}.$$

Then

$$0 \leq \varphi(x) = \inf\{d(x, y) \mid y \in W\} \leq \inf\{d(x, y_n) \mid y \in W\} = 0,$$

and hence $\varphi(x) = 0$.

3. Let $\varphi_1(x) = d(x, A_1)$ and $\varphi_2(x) = d(x, A_2)$. If $x_1 \in A_1$ and $x_2 \in A_2$, then clearly

$$\varphi_2(x_2) \leq d(x_1, x_2) \quad \text{and} \quad \varphi_1(x_1) \leq d(x_1, x_2).$$

We define the open sets

$$U_1(x_1) = \left\{ y \in S \;\middle|\; d(x_1, y) < \frac{1}{3}\,\varphi_2(x_1) \right\},$$

$$U_1 = \bigcup_{x_1 \in A_1} U_1(x_1) \quad \text{open}, \quad U_1 \supseteq A_1,$$

and

$$U_2(x_2) = \left\{ y \in S \;\middle|\; d(x_2, y) < \frac{1}{3}\,\varphi_1(x_2) \right\},$$

$$U_2 = \bigcup_{x_2 \in A_2} U_2(x_2) \quad \text{open}, \quad U_2 \supseteq A_2.$$

We shall prove that $U_1 \cap U_2 = \emptyset$.

Indirect proof. Assume that there exists $z \in U_1 \cap U_2$. Then there are an $x_1 \in A_1$ and an $x_2 \in A_2$, such that also

$$z \in U_1(x_1) \cap U_2(x_2).$$

Then we get the following contradiction,

$$0 < d(x_1, x_2) \le d(x_1, z) + d(z, x_2) \le \frac{1}{3}\,\varphi_1(x_2) + \frac{1}{3}\,\varphi_2(x_1) \le \frac{2}{3}\,d(x_1, x_2).$$

This cannot be true, so our assumption must be wrong. We therefore conclude that $U_1 \cap U_2 = \emptyset$, and the claim is proved.

Example 4.5 *Let $S = \{x \in \mathbb{R} \mid 0 \le x < 1\}$. Consider the family of subsets $\mathcal{T}$ in S consisting of the empty set $\emptyset$ and every subset $U \subseteq S$ of the form*

$$U = \{x \in \mathbb{R} \mid 0 \le x < k\}$$

for a number k with $0 < k \le 1$.

1. *Show that $\mathcal{T}$ is a topology on S.*
2. *Show that in the topological space $(S, \mathcal{T})$, the sequence $\left(x_n = \dfrac{1}{n+1} \right)$ will have every point in S as limit point.*
3. *Examine if the topology $\mathcal{T}$ stems from a metric on S.*

1. **TOP 1.** Let $U_i = \{x \in \mathbb{R} \mid 0 \le x < k_i\}$, $i \in I$. Then

   $$\bigcup_{i \in I} U_i = \{x \in \mathbb{R} \mid 0 \le x < \sup_{i \in I} k_i\} \in \mathcal{T},$$

   because

   $$\sup\{k_i \mid i \in I\} \in]0, 1].$$

   TOP 2. If $I = \{1, 2, \dots, n\}$, then

   $$\min_{k=1,\dots,n} k_i \in]0, 1],$$

   hence

   $$\bigcap_{i=1}^{n} U_i = \{x \in \mathbb{R} \mid 0 \le x < \min_{i=1,\dots,n} k_i\} \in \mathcal{T}.$$

   TOP 3. Finally, it is obvious that $\emptyset$, $S \in \mathcal{T}$.

   We have proved that $\mathcal{T}$ is a topology.

2. Let $x \in [0, 1[$. Then any open neighbourhood of x is of the form

   $$U = \{y \in \mathbb{R} \mid 0 \le y < k\}, \qquad \text{where } x < k.$$

   It follows from $x_n = \dfrac{1}{n+1} < k$ for $n > \dfrac{1}{k} - 1 = n_0$ that $x_n \to x$ for $n \to \infty$. Since $x \in S$ is chosen arbitrarily, we conclude that $x_n \to x$ for $n \to \infty$ for every $x \in S$ in $\mathcal{T}$.

3. The topology can never be generated of a metric. In fact, a metric space is automatically a Hausdorff space, and in a Hausdorff space S any sequence (x_n) has at most one limit point. In the present example every point of S is a limit point.

5 Sequences

Example 5.1 *Deduce the existence of a supremum from the principle of nested intervals.*

We *assume* that if

$$[a_1, b_1] \supseteq [a_2, b_2] \supseteq \cdots \supseteq [a_n, b_n] \supseteq \cdots$$

is a decreasing sequence of closed intervals, where the lengths of the intervals $|b_n - a_n| \to 0$ for $n \to \infty$, then the intersection $\bigcap_{n=1}^{\infty} [a_n, b_n]$ becomes just one number.

We shall *prove* that every non-empty bounded subset A of $\mathbb{R}$ has a smallest upper bound, $\sum A$.

Let $A \neq \emptyset$ be bounded, i.e. there exist a_1 and b_1, such that $A \subseteq [a_1, b_1]$.
Define $c_1 = \dfrac{1}{2}(a_1 + b_1)$ as the midpoint of the interval $[a_1, b_1]$.

1. If $x < c_1$ for every $x \in A$, then put

$$a_2 = a_1 \qquad \text{and} \qquad b_2 = c_1 .$$

2. If there exists an $x \in A$, such that $x \geq c_1$, we put

$$a_2 = c_1 \qquad \text{and} \qquad b_2 = b_1.$$

When this process is repeated, we obtain a decreasing sequence

$$[a_1, b_1] \supseteqq [a_2, b_2] \supseteqq \cdots \supseteqq [a_n, b_n] \supseteqq \cdots ,$$

of intervals, where

$$|b_n - a_n| = \frac{1}{2^{n-1}}\,|b_1 - a_1| \to 0 \qquad \text{for } n \to \infty,$$

hence by the assumption,

$$\bigcap_{n=1}^{\infty} [a_n, b_n] = \{x_0\}.$$

Furthermore, $a_n \nearrow x_0$ and $b_n \searrow x_0$. Since the construction secures that every b_n is an upper bound of A, thus

$$x \le b_n \qquad \text{for every } x \in A \text{ and every } n \in \mathbb{N},$$

we conclude that x_0 is also an upper bound of A.

Since none of the a_n is an upper bound of A, because we by the construction always can find an $x_n \in A$, such that $a_n < x_n$, and since $a_n \nearrow x_0$, we infer that x_0 is the smallest upper bound of A, hence $x_0 = \sup A$.

Example 5.2 *Let S be a topological space, and let (f_n), or in more detail $f_1, f_2, \dots, f_n, \dots$, be a sequence of continuous functions $f_n : S \to \mathbb{R}$, such that for all $x \in S$ it holds that*

(i) $f_n(x) \ge 0$,

(ii) $f_1(x) \ge f_2(x) \ge \cdots \ge f_n(x) \ge \cdots$,

(iii) $\lim_{n\to\infty} f_n(x) = 0$.

In other words: The decreasing sequence of functions (f_n) converges pointwise to the 0-function.
For $\varepsilon > 0$ and $n \in \mathbb{N}$ we set

$$U_n(\varepsilon) = \{x \in S \mid 0 \le f_n(x) < \varepsilon\}.$$

1. *Show that $U_n(\varepsilon)$ is an open set in S.*
2. *Show that for fixed $\varepsilon > 0$, the collection of sets $\{U_n(\varepsilon) \mid n \in \mathbb{N}\}$ defines an open covering of S.*
3. *Now assume that S is compact. Show then that for every $\varepsilon > 0$ there is an $n_0 \in \mathbb{N}$, such that for all $n \ge n_0$ it holds that*

   $$0 \le f_n(x) < \varepsilon \qquad \textit{for all } x \in S;$$

   or written with quantifiers,

   $$\forall \varepsilon > 0 \exists n_0 \in \mathbb{N} \forall n \in \mathbb{N} : n \ge n_0 \quad \Longrightarrow \quad \forall x \in S : 0 \le f_n(x) < \varepsilon.$$

   We conclude that under the given assumptions, the sequence of functions (f_n) converges uniformly to the 0-function.
   This result is due to the Italian mathematician Ulisse Dini (1845 – 1918) and is known as Dini's Theorem.
4. *Is it of importance that S is compact in (3)?*

1. Since every $f_n \ge 0$, and each f_n is continuous, we get

   $$U_n(\varepsilon) = \{x \in S \mid 0 \le f_n(x) < \varepsilon\} = f_n^{\circ -1}(]-\infty, \varepsilon[) \quad \text{open.}$$

2. Since

   $$f_1(x) \ge f_2(x) \ge \cdots \ge f_n(x) \ge \cdots \to 0,$$

   there is to every $\varepsilon > 0$ and every $x \in S$ an $n \in \mathbb{N}$, such that $0 \le f_n(x) < \varepsilon$, i.e. $x \in U_n(\varepsilon)$. Since this holds for every $x \in S$, we have

   $$S \subseteqq \bigcup_{n=1}^{\infty} U_n(\varepsilon),$$

   so $\{U_n(\varepsilon) \mid n \in \mathbb{N}\}$ is an open covering of S, because every $U_n(\varepsilon)$ is an open set.

3. If S is compact, then the open covering $\{U_n(\varepsilon) \mid n \in \mathbb{N}\}$ of S can be thinned out to a finite covering,

$$S \subseteqq U_{n_1}(\varepsilon) \cup U_{n_2}(\varepsilon) \cup \cdots \cup U_{n_k}(\varepsilon).$$

It remains to notice that if $x \in U_{n_0}(\varepsilon)$, then $x \in U_n(\varepsilon)$ for every $n \geq n_0$, hence

$$U_{n_0}(\varepsilon) \subseteqq U_{n_0+1}(\varepsilon) \subseteqq \cdots .$$

This follows from $f_{n+1}(x) \leq f_n(x) < \varepsilon$. Then we get for $n_1 < n_2 < \cdots < n_k$,

$$S = U_{n_1}(\varepsilon) \cup U_{n_2} \cup \cdots \cup U_{n_k}(\varepsilon) = U_{n_k}(\varepsilon),$$

hence

$$S = \{x \in S \mid f_{n_k}(x) < \varepsilon\}.$$

If $n \geq n_k$, then it follows that

$$0 \leq f_n(x) \leq f_{n_k}(x) < \varepsilon,$$

hence

$$0 \leq f_n(x) < \varepsilon \qquad \text{for } n \geq n_k,$$

and we have proved Dini's theorem.

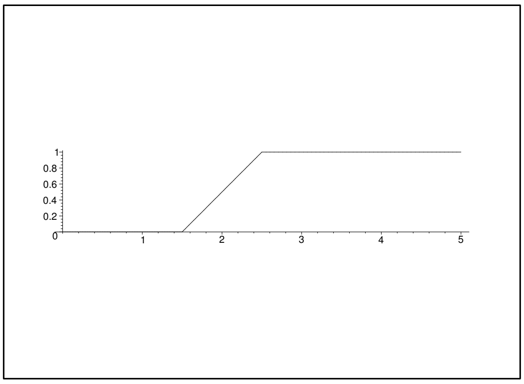


Figure 15: A principal sketch of the graph of f_n.

4. The assumption of compactness is of course important. In order to see this, we must construct an example, in which S is not compact, where the $f_n \geq 0$ are all continuous and tend pointwise and decreasingly towards 0, and where the convergence is not uniform.

Consider $S = [0, \infty[$, which clearly is not compact. We put

$$f_n(x) = \begin{cases} 0, & \text{for } x \in [0, n-1[, \\ x - n + 1, & \text{for } x \in [n-1, n[, \\ 1, & \text{for } x \in [n, \infty[, \end{cases} \qquad n \in \mathbb{N}.$$

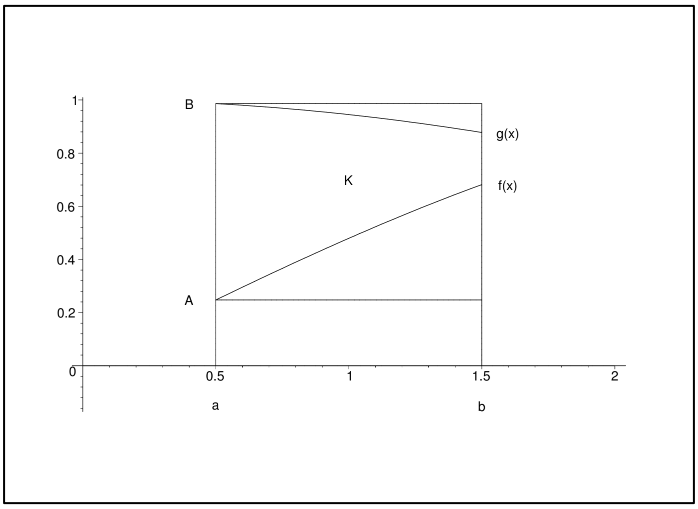
Then every $f_n \geq 0$ is continuous and $f_n(x) \searrow 0$ for $n \to \infty$ for every $x \in S$, so the convergence is decreasing. Also, to every $n_0 \in \mathbb{N}$ there exist an $n \geq n_0$ and an $x \in S$, such that $f_n(x) = 1$. This holds for all $n \geq n_0$ and all $x \geq n$, and the convergence is not uniform.

Remark 5.1 The example above illustrates the common observation in Mathematics, that if something can go wrong, it can go really wrong!. $\Diamond$

Example 5.3 *Let $f,\ g:[a,b]\to\mathbb{R}$ be continuous functions defined in a closed and bounded interval $[a,b]$. Assume that $f(x)<g(x)$ for every $x\in[a,b]$.*
Show that

$$K=\{(x,y)\in\mathbb{R}^2\mid a\le x\le b,\ f(x)\le y\le g(x)\}$$

is a compact subset in $\mathbb{R}^2$.



Since $[a,b]$ is compact, and f and g are continuous, f has a minimum, $f(x_1)=A$, and g a maximum $g(x_2)=B$, and we infer that

$$K=\{(x,y)\in\mathbb{R}^2\mid a\le x\le b,\ f(x)\le y\le g(x)\}\subseteqq[a,b]\times[A,B],$$

proving that K is bounded.

Furthermore, K is closed. This is proved by showing that the complementary set is open.
There is nothing to prove if $(x,y)\in\mathbb{R}^2\setminus K$ satisfies one of the following conditions,

$$i)\ \ x<a,\quad ii)\ \ x>b,\quad iii)\ \ y<A,\quad iv)\ \ y>B.$$

Let

$$(x_0,y_0)\in([a,b]\times[A,B])\setminus K.$$

We may assume that $y_0<f(x_0)$, because the other cases are treated analogously. Using that f is continuous we can to

$$\varepsilon=\frac{1}{3}\,\{f(x_0)-y_0\}>0$$

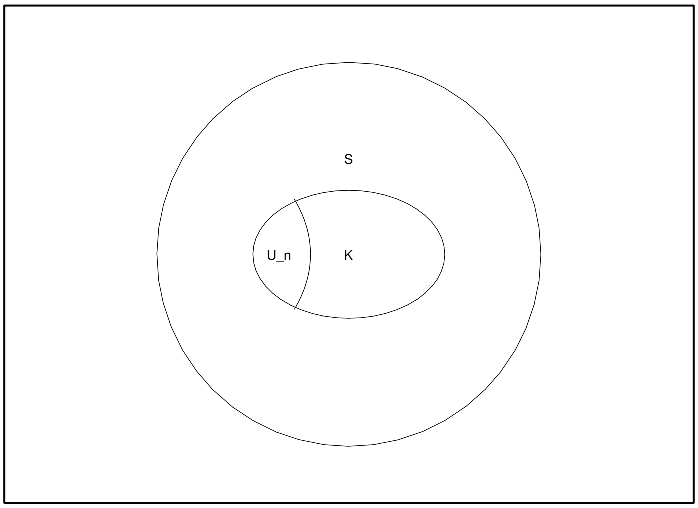
find a $\delta>0$, such that

$$|f(x)-f(x_0)|<\varepsilon\qquad\text{for } |x-x_0|<\delta \text{ and } x\in[a,b].$$

Then $]x_0-\delta,x_0+\delta[\,\times\,]y_0-\varepsilon,y_0+\varepsilon[$ and K are disjoint sets, hence the complementary set of K is open. This implies that K is closed.

We have proved that K is closed and bounded in$\mathbb{R}^2$, so K is compact.

Example 5.4 *Let $K_1 \supseteqq K_2 \supseteqq \cdots \supseteqq K_n \supseteqq \cdots$ be a descending sequence of non-empty subsets in a Hausdorff space S. Show that the intersection of sets $\bigcap_{n=1}^{\infty} K_n$ is non-empty.*



Indirect proof. *Assume* that $\bigcap_{n=1} \infty K_n = \emptyset$, and consider the subspace topology $\mathcal{T}_K$ of $K = K_1$. Then

$$U_n = K \setminus K_n \qquad \text{åben i } \mathcal{T}_K.$$

We have

$$\bigcup_{n=1}^{\infty} U_n = \bigcup_{n=1}^{\infty} (K \setminus K_n) = K \setminus \bigcap_{n=1}^{\infty} K_n = K,$$

where we in the latter equality have applied the assumption that $\bigcap_{n=1}^{\infty} K_n = \emptyset$. Since K is compact in a Hausdorff space, and $\bigcup_{n=1}^{\infty} U_n$ is an open covering, and

$$\emptyset = U_1 \subseteqq U_2 \subseteqq \cdots \subseteqq U_n \subseteqq \cdots ,$$

we can extract from this covering a finite covering (with $n_1 < n_2 < \cdots < n_k$),

$$K = K_1 \subseteqq \bigcup_{j=1}^{k} U_{n_j} = U_{n_k} = K \setminus K_{n_k}.$$

Now, $K_{n_k} \subseteqq K$ and $K = K \setminus K_{n_k}$, so $K_{n_k} = \emptyset$ contradicting the assumption that none of the K_n is empty.

Hence our assumption is wrong, and we infer that

$$\bigcap_{n=1}^{\infty} K_n \neq \emptyset.$$

Example 5.5 *Let $S = \mathbb{N} \cup \{0\}$ be the set of non-negative integers. For every natural number $n \in \mathbb{N}$ we define a subset U_n in S by*

$$U_n = \{n \cdot p \in S \mid p = 0,\, 1,\, 2,\, \dots\}.$$

1. *Show that for all n, $m \in \mathbb{N}$, the intersection $U_n \cap U_m$ has the form U_k for a suitable $k \in \mathbb{N}$.*

Consider the family $\mathcal{T}$ of subsets in S which consists of the empty set $\emptyset$ and all subsets U in S that can be written as a union of sets from $\{U_n \mid n \in \mathbb{N}\}$, i.e.

$$U = \bigcup_{\alpha \in A} U_{n_\alpha}.$$

2. *Show that $\mathcal{T}$ is a topology on S.*
(The system of subsets $\{U_n \mid n \in \mathbb{N}\}$ in S is called a basis for the topology $\mathcal{T}$.)

3. *Show that the sequence $(x_n = n!)$ will converge to every point in the topological space $(S, \mathcal{T})$.*

1. Let $k \in \mathbb{N}$ be the smallest number which can be divided by both m and n. Then we get

$$U_n \cap U_m = U_k.$$

2. The result of **1.** implies that finite intersections of sets of the type U_n again can be written as an U_k. When we form the topology by adding any union of sets of type U_k as open sets, supplied by $\emptyset$, it only remains to note that the whole space $S = U_1$ trivially belongs to $\mathcal{T}$. This proves that $\mathcal{T}$ is a topology.

3. Let $y_0 \in \mathbb{N}$. Then the smallest open set, which contains y_0, must necessarily be U_{y_0}.
If we choose $n_0 \in \mathbb{N}$, such that $n_0! = y_0 \cdot k$ for some $k \in \mathbb{N}$, it follows that $n! \in U_{y_0}$ for every $n \geq n_0$.

   If instead $y_0 = 0 \in S$, then every U_k is a neighbourhood. Choose $n_0 \in \mathbb{N}$, such that $n_0! = k \cdot p$, $p \in \mathbb{N}$, and we obtain that $n! \in U_k$ for $n \geq n_0$.

   This implies by the definition of convergence in topological spaces that $(n!)$ converges towards every point in S.

6 Semi-continuity

Example 6.1 *Let S be a Hausdorff space. A function $f : S \to \mathbb{R}$ is said to be* lower semi-continuous, *if the following condition is satisfied:*
For every $x \in S$ and every $\varepsilon > 0$ there exists a neighbourhood N and an x in S, such that

$$f(x) - \varepsilon < f(y) \qquad \text{for } y \in N.$$

1. *Show that a lower semi-continuous function $f : S \to \mathbb{R}$ is bounded from below on every sequentially compact subset K in S.*
   HINT: You can prove this indirectly.

2. *Show that a lower semi-continuous function $f : S \to \mathbb{R}$ assumes a minimum value on every sequentially compact subset K in S.*
   HINT: Construct a sequence (x_n) on K for which

   $$\lim_{n\to\infty} f(x_n) = \inf f(K),$$

   and make use of this to determine a point $x_0 \in K$, such that

   $$f(x_0) = \inf f(K).$$

   (In a suitable setting this is the so-called *direct method* the calculus of variations, and the sequence (x_n) is called a minimizing sequence.)

1. *Indirect proof.* Let K be sequentially compact. We assume that f is *not* bounded from below on K. This means that we can find points $x_n \in K$, such that $f(x_n) < -n$, $n \in \mathbb{N}$. We may assume that all $(x_n) \subseteq K$ are mutually different.

   Since K is sequentially compact, (x_n) has an accumulation point $x_0 \in K$.

   Since f is lower semi-continuous in x_0, we can to every $\varepsilon > 0$ find a neighbourhood N of x_0 in S, such that

   $$f(x_0) - \varepsilon < f(y) \qquad \text{for all } y \in N.$$

   Since x_0 is an accumulation point, there are (infinitely many) $x_n \in N$ for which

   $$-n < f(x_0) - \varepsilon.$$

   Since also $x_n \in N$, it follows that

   $$f(x_n) < -n < f(x_0) - \varepsilon < f(x_n),$$

   which is a contradiction.

   We have proved that every lower semi-continuous function $f : S \to \mathbb{R}$ is bounded from below on every sequentially compact set.

2. We infer from the definition of $\inf f(K)$ that there exists a sequence $(x_n) \subseteq K$, such that

   $$\lim_{n\to\infty} f(x_n) = \inf f(K).$$

   The sequence (x_n) itself needs not be convergent, but since it has an accumulation point $x_0 \in K$, we can extract from it an subsequence which converges towards x_0. The image of the sequence will still converge towards $\inf f(K)$, so we may already from the beginning assume that $(x_n) \to x_0$.

   By assumption, f is lower semi-continuous in x_0, so to every $\varepsilon > 0$ there is a neighbourhood N of x_0, such that

   $$f(x_0) - \varepsilon < f(y) \qquad \text{for every } y \in N.$$

   Using that N is a neighbourhood of x_0 and that $x_n \to x_0$ for $n \to \infty$, we infer that there exists an m, such that $x_n \in N$ for all $n \geq m$, thus

   $$f(x_0) - \varepsilon < f(x_n) \qquad \text{for all } n \geq m.$$

   The right hand side is convergent for $n \to \infty$, hence

   $$f(x_0) - \varepsilon \leq \lim_{n\to\infty} f(x_0) = \inf f(K) \quad \text{for every} \varepsilon > 0.$$

   Finally, we get by taking the limit $\varepsilon \searrow 0$,

   $$\inf f(K) \leq f(x_0) \leq \inf f(K),$$

   proving that

   $$f(x_0) = \inf f(K) \qquad [= \min f(K)].$$

Example 6.2 *Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function with bounded differential quotient. Show that f is uniformly continuous.*
Hint. You can use the classical Mean Value Theorem.

The differential quotient $Df(z)$ is bounded, so

$$\sup_{z\in\mathbb{R}} |Df(x)| = C < +\infty.$$

Using the Mean Value Theorem we get for any x and y that there exists an z between x and y, such that

$$f(y) - f(x) = Df(z) \cdot (y - z), \qquad z = z(z, y),$$

hence

$$|f(y) - f(x)| = |Df(z)| \cdot |y - x| \le C \cdot |y - x|.$$

To any $\varepsilon > 0$ we choose $\delta = \dfrac{\varepsilon}{C}$, such that

$$|y - x| < \delta \qquad \text{implies that} \quad |f(y) - f(x)| < \varepsilon,$$

where δ is independent of x and y. This means that f is uniformly continuous.

Example 6.3 *A subset K in a metric space (S, d) is called* precompact *if for every $\varepsilon > 0$ there exist finitely many points $x_1, \dots, x_p \in K$ such that*

$$K \subseteqq B_\varepsilon(x_1) \cup \cdots \cup B_\varepsilon(x_p).$$

1. *Show that a subset $K \subseteqq \mathbb{R}^n$ in the space $\mathbb{R}^n$ (with Euclidean metric) is precompact if and only if it is bounded.*
2. *Let $f : X \to Y$ be a mapping between the metric spaces (X, d_X) and (Y, d_Y), and let $K \subseteqq X$ be a precompact subset in X. Show that if f is uniformly continuous in K, then the image set $f(K) \subseteqq Y$ is precompact in Y.*

1. Let K be precompact, and put $\varepsilon = 1$. There are points $x_1, \dots, x_p$, such that

$$K \subseteqq B_1(x_1) \cup \cdots \cup B_1(x_p).$$

Defining

$$R = \max\{d(x_1, x_j) \mid j = 1, \dots, p\} + 1,$$

it follows that $B_1(x_j) \subseteqq B_R(x_1)$ for every $j \in \{1, \dots, p\}$, thus

$$K \subseteqq B_1(x_1) \cup \cdots \cup B_1(x_p) \subseteqq B_R(x_1),$$

and K is bounded.

The phrase "K bounded" means that "K can be shut up" in a ball. Therefore, in order to prove the claim in the opposite direction it suffices to show that whenever a ball $K_R(x_0)$ and an $\varepsilon > 0$ are given, then there exist finitely many points $x_1, \dots, x_p \in S$, such that

$$B_R(x_0) \subseteqq B_\varepsilon(x_1) \cup B_\varepsilon(x_2) \cup \cdots \cup B_\varepsilon(x_p).$$

Remark 6.1 It is at this point that we use that the metric is *Euclidean*. In general, the claim is wrong for metric spaces, which is illustrated by the following example.

Let $X = \mathbb{R}$ be equipped with the metric

$$d(x,y) = \begin{cases} 1 & \text{for } x \neq y, \\ 0 & \text{for } x = y. \end{cases}$$

A routine check shows that this is indeed a metric. Then $\mathbb{R} \subseteqq B_1(0)$ is clearly bounded (the radius is 1).
If $0 < \varepsilon < 1$, then $B_\varepsilon(x) = \{x\}$, and

$$\mathbb{R} = \bigcup_{x \in \mathbb{R}} \{x\}$$

is obviously not precompact.

This example shows that we must require more of the metric – it is quite natural her to assume that it is Euclidean. $\Diamond$

We consider $\mathbb{R}^2$ with the usual Euclidean metric. Choose any $\varepsilon > 0$, and assume that

$$K \subseteqq [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n].$$

Each edge $[a_j, b_j]$ can be divided into at most

$$\frac{\sqrt{n}}{2\varepsilon} \, |B - j - a_j| + 1 \text{ intervals of length } \frac{2\varepsilon}{\sqrt{n}}.$$

This implies that

$$M \leq \frac{1}{(2\varepsilon)^n} \, (\sqrt{n})^n \prod_{j=1}^{n} \{|b_j - a_j| + 1\}$$

n-dimensional cubes cover K. Choose the centre of each of these cubes as centre of balls of radius ε. Then every cube is again covered by a finite number of balls, and the claim follows.

2. If f is uniformly continuous on a precompact set K, then $\delta = \delta(\varepsilon)$ depends only on $\varepsilon > 0$ and not on x, $y \in K$. Hence, if

$$d_X(x,y) < \delta(\varepsilon), \qquad \text{then} \qquad d_Y(f(x), f(y)) < \varepsilon,$$

and thus

$$(2) \qquad f\left(B_{X,\delta}(x)\right) \subseteqq B_{Y,\varepsilon}(f(x)).$$

The set K is precompact, so there exist $x_1, \dots, x_p \in K$, such that

$$K \subseteqq B_{X,\delta}(x_1) \cup \cdots \cup B_{X,\delta}(x_p).$$

It follows from (2) that

$$\begin{aligned} f(K) &\subseteqq f\left(B_{X,\delta}(x_1)\right) \cup \cdots \cup f\left(B_{X,\delta}(x_p)\right) \\ &\subseteqq B_{Y,\varepsilon}\left(f(x_1)\right) \cup \cdots \cup B_{Y,\varepsilon}\left(f(x_p)\right). \end{aligned}$$

Since this holds for every $\varepsilon > 0$, we conclude that $f(K)$ is precompact.

Example 6.4 *Let T be a point set with more that one element equipped with the discrete topology.*

1. *Show that a topological space S is connected if and only if every continuous mapping $f : S \to T$ is constant.*

2. *Let $\{W_i \mid i \in I\}$ be a family of connected subsets in a topological space S, such that for every pair of sets W_i and W_j from the family it holds that $M_i \cap W_j \neq \emptyset$. Then show that the union $\bigcup_{i \in I} W_i$ is a connected subset in S.*

1. Let $f : S \to T$ be a continuous function, which is not constant, with e.g. $\{t_1, t_2\} \subseteq f(S)$. Since $\{t_1\}$ and $\{t_2\}$ are open, both $f^{\circ -1}(\{t_1\})$ and $f^{\circ -1}(\{t_2\})$ are open, disjoint and non-empty, and S is not connected.

   Hence we get by contraposition that if S is connected, then every continuous function $f : S \to T$ is constant.

   If $S = S_1 \cup S_2$ is not connected, i.e. S_1 and S_2 are open, non-empty and disjoint, then we can define a continuous function $f : S \to T$ by

$$f(x) = \begin{cases} t_1, & \text{for } x \in S_1, \\ t_2, & \text{for } x \in S_2. \end{cases}$$

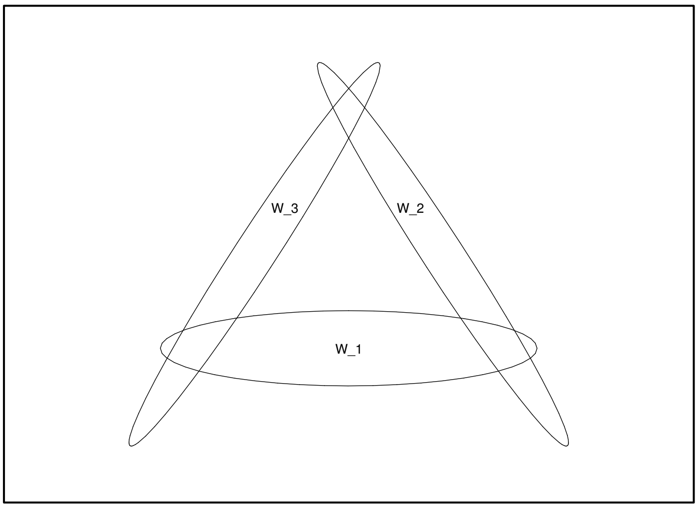
In fact,

$$f^{\circ -1}(\{t_1\}) = S_1 \qquad \text{and} \qquad f^{circ-1}(\{t_2\}) = S_2,$$

and

$$f^{\circ -1}(U) = \emptyset \qquad \text{for } U \subsetneqq S \setminus \{t_1, t_2\}.$$

Clearly, f is not constant, and the claim is proved.



2. Since $f_i : W_i \to T$ is continuous, so

$$f_i(x) = t_i \in T \qquad \text{for } x \in W_i,$$

and we infer that if

$$f : \bigcup_{i \in I} W_i \to T \qquad \text{is continuous,}$$

then $f(x) = f_i(x) = t_i$ for $x \in W_i$, $i \in I$.
Since there is an $x \in W_i \cap W_j$, we must have

$$f(x) = f_i(x) = t_i = f_j(x) = t_j,$$

thus $t_i = t_j$ for all i, $j \in I$. This implies that the constant functions

$$f : \bigcup_{i \in I} W_i \to T$$

are the only continuous functions, and we conclude from **1.** that $\bigcup_{i \in I} W_i$ is connected.

Example 6.5 *Prove the following theorem: Let M be an arbitrary subset in the number space $\mathbb{R}^k$ with the usual topology, and let $\{U_i \mid i \in I\}$ be an arbitrary system of open sets in $\mathbb{R}^k$ that covers M. Then, either there exists a finite subsystem $\{U_{i_1}, \dots, U_{i_n}\}$, or, there exists a countable subsystem $\{U_{i_1}, U_{i_2}, \dots, U_{i_n}, \dots\}$ that covers M.*
The theorem is due to the Finnish mathematician Ernst Lindelöf (1870–1946), and it is called Lindelöf's covering theorem.
HINT*: consider the following system of open balls in $\mathbb{R}^k$:*

$$B_r(x) \mid r \in \mathbb{R}^+ \text{ is rational}; x \in \mathbb{R}^k \text{ has rational coordinates}\}.$$

We mention without proof that this system is countable.

Once the hint is given the example becomes extremely simple. In fact,

$$\mathbb{R} \subseteqq \bigcup \left\{B_r(x) \mid r \in \ QQ^+;\, x \in \mathbb{Q}^k\right\} = \bigcup_{n=1}^{\infty} B_n,$$

where $\mathbb{Q}$ denotes the set of rational numbers.

Each element U_i, $i \in I$, can as an open set be written

$$U_i = \bigcup_{n \in I_i} B_n, \qquad \text{where } I_i \subseteqq \mathbb{N}, \text{ thus } I_i \text{ is countable.}$$

Since $\{U_i \mid i \in I\}$ covers M, there exists a subsystem $\{B_n \mid n \in J\}$, $J \subseteqq \mathbb{N}$, which also covers M, (e.g. $J = \bigcup_{i \in I} I_i$).

The subsystem $\{B_n \mid n \in J\}$ is finite or countable, and every B_n is contained in some U_{i_n} for $n \in J$. Hence, we choose $\{U_{i_n} \mid n \in J\}$, such that

$$M \subseteqq \bigcup_{n \in J} B_n \subseteqq \bigcup_{n \in J} U_{i_n},$$

(finite or countable union).

7 Connected sets, differentiation a.o.

Example 7.1 *Let E be a subset in the topological space S. Show that if E is connected, then the closure $\overline{E}$ is also connected.*

Let T contain at least two points, and let T be equipped with the discrete topology, thus every point $\{t\} \subset T$ is both open and closed.

Let $\varphi : S \to T$ be a continuous mapping. Since $\varphi_{|E} : E \to T$ is continuous and E is connected, we conclude that $\varphi(x) = t_0 \in T$, $x \in E$ is constant on E.

We get $\overline{E}$ by adding the boundary ∂E to E, thus $\overline{E}$ is the set of all contact points. Therefore, $\varphi(x) = t_0 \in T$ is constant on $\overline{E}$, because we get the values on ∂E by continuous extension, i.e.

$$x_n \to x_0 \in \overline{E} \quad \text{implies} \quad \varphi(x_0) = \lim \varphi(x_n) = \lim t_0 = t_0.$$

Example 7.2 *Let (S, d) be a metric space. For an arbitrary pair of non-empty subsets A and B in S, we define the* distance *from A to B, denoted $\mathrm{d}(A, B)$, by*

$$\mathrm{d}(A, B) = \inf\{\mathrm{d}(x, y) \mid x \in A,\, y \in B\}.$$

1. *As in* Example 4.4 *we define for every $x \in S$ the distance from x to B by*

$$\mathrm{d}(x, B) = \inf\{\mathrm{d}(x, y) \mid y \in B\}.$$

Argue that for arbitrary points $x \in A$ and $y \in B$ it holds that

$$\mathrm{d}(A, B) \le \mathrm{d}(x, B) \quad \textit{and} \quad \inf\{\mathrm{d}(x', B) \mid x' \in A\} \le \mathrm{d}(x, y).$$

Utilize this to show that

$$\mathrm{d}(A, B) = \inf\{\mathrm{d}(x, B) \mid x \in A\}.$$

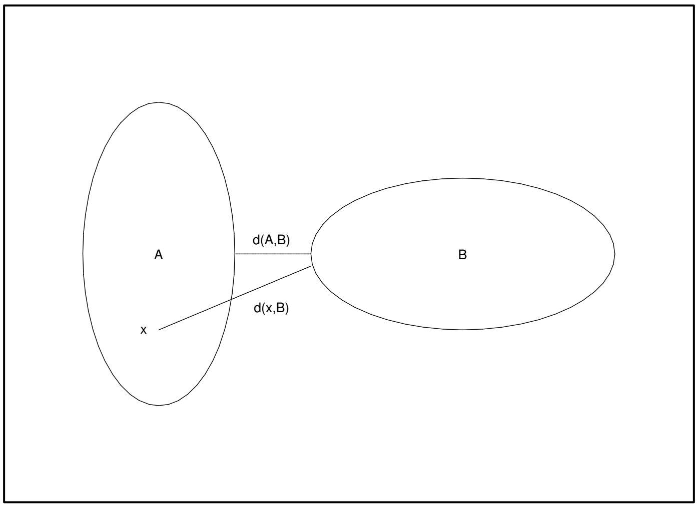
2. *Show that if A is compact, then there exists a point $a_0 \in A$ such that*

$$\mathrm{d}(A, B) = \mathrm{d}(a_0, B).$$

Hint: You can use that the function $\varphi : S \to \mathbb{R}$, defined by $\varphi(x) = d(x, B)$, is continuous.
Next show that if B is also compact, then there exists a point $b_0 \in B$ such that

$$\mathrm{d}(A, B) = \mathrm{d}(a_0, b_0).$$

3. *Let K be a compact subset in S contained in the open set U in S, i.e. $K \subseteqq U \subseteqq S$. Show that there exists an $r \in \mathbb{R}^+$, such that $B_r(x) \subseteqq U$ for every $x \in K$.*



1. It follows from $d(x, B) = \inf\{d(x, y) \mid y \in B\}$, that

$$d(x, B) = d(\{x\}, B) \ge d(A, B) \qquad \text{for every } x \in A,$$

and the claim is proved.
Furthermore,

$$d(x, B) = \inf\{d(x, y) \mid y \in B\} \le d(x, y_0), \qquad \text{for } y_0 \in B,$$

so

$$\inf\{d(x', B) \mid x' \in A\} \le d(x, B) \le d(x, y) \qquad \text{for } x \in A \text{ and } y \in B.$$

It follows from these two inequalities that

$$\begin{aligned}\inf\{d(x',B) \mid x' \in A\} &\leq \inf\{d(x,y) \mid x \in A,\, y \in B\}\\ &= d(A,B) \leq \inf\{d(x,B) \mid x \in A\},\end{aligned}$$

hence we have equality

$$d(A,B) = \inf\{d(x,B) \mid x \in A\}.$$

2. Now, $\varphi : S \to \mathbb{R}$, given by $\varphi(x) = d(x,B)$, is continuous, and A is compact. Therefore, $\varphi(x)$ attains its minimum at a point $a_0 \in A$, so

$$\varphi(a_0) = d(x_0,B) = \inf\{d(x,B) \mid x \in A\} = d(A,B).$$

If also B is compact, then use that $\psi(y) = d(a_0,y)$ is continuous, so there exists a $b_0 \in B$, such that

$$d(a_0,b_0) = \inf\{d(a_0,y) \mid y \in B\} = \inf\{d(x,y) \mid x \in A,\, y \in B\} = d(A,B).$$

3. It follows from $K \subseteqq U$ that $K \cap (S \setminus U) = \emptyset$. The mapping

$$\varphi(x) = d(x, S \setminus U)$$

is continuous, and since K is compact, we can by (2) find a point $x_0 \in K$, such that

$$d(K, S \setminus U) = d(x_0+, S \setminus U) > 0,$$

because $x_0 \notin S\setminus U$. (Strictly speaking we shall choose $R > 0$, such that $B = (S\setminus U) \cup \overline{B_R(x_0)} \neq \emptyset$, thus B is closed and bounded, etc.)
Then choose $r \in]0, d(K, S \setminus U)[$, and we have

$$B_r(x) \cup S \setminus U = \emptyset \qquad \text{for all } x \in K,$$

hence

$$B_r(x) \subseteqq U \qquad \text{for all } x \in K.$$

Example 7.3 *Let $E = C^\infty([0,2\pi],\mathbb{R})$ be the vector space of differentiable functions $f : [0,2\pi] \to \mathbb{R}$ of class C^∞. For $f \in E$ we set*

$$\|f\|_0 = \sup\{|f(x)| \mid x \in [0,2\pi]\},$$

$$\|f\|_1 = \sup\{|f(x)| + |f'(x)| \mid x \in [0,2\pi]\}.$$

1. *Show that $\|\cdot\|_0$ and $\|\cdot\|_1$ are norms in E.*

Define the linear mapping $D : E \to E$ by associating to $f \in E$ the derivative $f' \in E$ of f, i.e.

$$D(f) = f' \qquad \text{for } f \in E.$$

2. *Show that for every $n \in \mathbb{N}$ there exists a function $f_n \in E$ for which $\|f_n\|_0 = 0$ and $\|D(f_n)\|_0 = n$. Utilize this to show that $D : E \to E$ is not continuous, when E is equipped with the norm $\|\cdot\|_0$.*

3. *Show that $D : E_1 \to E_0$ is continuous, when E_1 is E equipped with the norm $\|\cdot\|_1$ and E_0 is E equipped with the norm $\|\cdot\|_0$.*

1. Obviously, $\|f\|_0 \geq 0$, and if

$$\|f\|_0 = \sup\{|f(x)| \mid x \in [0,2\pi]\} = 0,$$

then $f(x) = 0$ for every $x \in [0,2\pi]$, thus $f = 0$.
Furthermore,

$$\|\alpha f\|_0 = \sup\{|\alpha f(x)| \mid x \in [0,2\pi]\} = |\alpha| \sup\{|f(x)| \mid x \in [0,2\pi]\} = |\alpha|\|f\|_0.$$

Finally, we get concerning the triangle inequality

$$\begin{aligned}\|f+g\|_0 &= \sup\{|f(x)+g(x)\| \mid x \in [0,2\pi]\} \leq \sup\{|f(x)| + |g(x)| \mid x \in [0,2\pi]\} \\ &\leq \sup\{|f(x)| \mid x \in [0,2\pi]\} + \sup\{|g(x)| \mid x \in [0,2\pi]\} = \|f\|_0 + \|g\|_0.\end{aligned}$$

Summing up we have proved that $\|\cdot\|_0$ is a norm.

Then $\|f\|_1 \geq \|f\|_0 \geq 0$. If

$$\|f\|_1 = \sup\{|f(x)| + |f'(x)| \mid x \in [0,2\pi]\} = 0,$$

then

$$|f(x)| + |f'(x)| = 0 \qquad \text{for every } x \in [0,2\pi].$$

This implies that $\|f(x)| = 0$ for every $x \in [0,2\pi]$, i.e. $f = 0$.
Furthermore,

$$\|\alpha f\|_1 = \sup\{|\alpha f(x)|+|\alpha f'(x)| \mid x \in [0,2\pi]\} = |\alpha| \sup\{|f(x)|+|f'(x)| \mid x \in [0,2\pi]\} = |\alpha|\cdot\|f\|_1.\}$$

Finally,

$$\begin{aligned}\|f+g\|_1 &= \sup\{|f(x)+g(x)| + |f'(x)+g'(x)| \mid x \in [0,2\pi]\} \\ &\leq \sup\{|f(x)| + |f'(x)| \mid x \in [0,2\pi]\} + \sup\{|g(x)| + |g'(x)| \mid x \in [0,2\pi]\} = \|f\|_1 + \|g\|_1.\end{aligned}$$

We conclude that $\|.\|_1$ is also a norm.

2. The form of the interval $[0,2\pi]$ indicates that we shall think of trigonometric examples. Choosing

$$f_n(x) = \sin nx, \qquad x \in [0,2\pi],$$

it follows that $f_n \in E$ and

$$f_n'(x) = n \cdot \cos nx, \qquad x \in [0,2\pi],$$

thus

$$\|f_n\|_0 = \sup\{|\sin nx| \mid x \in [0, 2\pi]\} = 1,$$

and

$$\|D(f_n)\|_0 = \sup\{n|\cos nx| \mid x \in [0, 2\pi]\} = n.$$

Clearly, the sequence $g_n = \frac{1}{n} f_n$ converges towards 0, because

$$\|g_n\|_0 = \frac{1}{n} \to 0 \qquad \text{for } n \to \infty.$$

The image sequence $\|D(g_n)\|_0 = \frac{1}{n} \cdot n = 1$ does *not* converge towards 0, and $D : E_0 \to E_0$ is not continuous.

3. The claim follows from the estimate

$$\begin{aligned} \|D(f)\|_0 &= \sup\{|f'(x)| \mid x \in [0, 2\pi]\} \\ &\le \sup\{|f(x)| + |f'(x)| \mid x \in [0, 2\pi]\} = \|f\|_1. \end{aligned}$$

The mapping D is linear, so it suffices to prove the continuity at 0:
To any given $\varepsilon > 0$ we choose $\delta = \varepsilon > 0$. If $\|f\|_1 < \delta = \varepsilon$, then $\|D(f)\|_0 \le \|f\|_1 < \varepsilon$, and we have proved that $D : E_1 \to E_0$ is continuous.

Remark 7.1 The example shows that the same mapping $D : E \to E$ can be continuous in one topology and discontinuous in another one. Both norms $\|\|_0$ and $\|\cdot\|_1$ are classically known. $\Diamond$

8 Addition and multiplication by scalars in normed vector spaces

Example 8.1 *Let V be the space of continuous functions $f : \mathbb{R} \to \mathbb{R}$, such that $f(x) \to 0$ for $|x| \to \infty$. For a function $f \in V$ holds, in other words*

$$\forall \varepsilon > 0 \exists a \in \mathbb{R}^{+} \forall x \in \mathbb{R} : |x| > a \quad \Longrightarrow \quad |f(x)| < \varepsilon.$$

Define the operations 'addition' and 'multiplication with scalars' in V by the obvious pointwise definitions.

1. *Show that V is a vector space.*

2. *Show that every function $f \in V$ is bounded.*

Making use of (2), we can define

$$\|f\| = \sup\{|f(x)| \mid x \in \mathbb{R}\} \qquad \text{for } f \in V.$$

3. *Show that $\| \cdot \|$ is a norm in V.*

1. If f, $g \in V$ and $\alpha \in \mathbb{R}$, then $f + \alpha \cdot g : \mathbb{R} \to \mathbb{R}$ is continuous, and

   $$f(x) + \alpha \cdot g(x) \to 0 + \alpha \cdot 0 = 0 \qquad \text{for } |x| \to \infty,$$

   so $f + \alpha \cdot g \in V$, and V is a vector space.

2. Let $f \in V$, and choose e.g. $\varepsilon = 1$. There exists an $a > 0$, such that $|f(x)| < 1$, whenever $|x| > a$. The residual set $[-a, a]$ is a compact interval. The function f is continuous, so $|f|$ has a maximum A on $[-a, a]$. Then

   $$\|f\| = \sup\{|f(x)| \mid x \in \mathbb{R}\} \leq \max\{1, A\} < \infty,$$

   and we have proved that f is bounded.

3. It follows from (2) that $\|f\| < \infty$ for every $f \in V$. The rest is well-known: $\|f\| \geq 0$, and if

   $$\|f\| = \sup\{|f(x)| \mid x \in \mathbb{R}\} = 0,$$

   then $f(x) = 0$ for every $x \in \mathbb{R}$, hence $f = 0$.
   Furthermore,

   $$\|\alpha \cdot f\| = \sup\{|\alpha\, f(x)\| \mid x \in \mathbb{R}\} = |\alpha| \sup\{|f(x)| \mid x \in \mathbb{R}\} = |\alpha| \cdot \|f\|,$$

   and

   $$\begin{aligned} \|f+g\| &= \sup\{|f(x)+g(x)| \mid x \in \mathbb{R}\} \leq \sup\{|f(x)|+|g(x)| \mid x \in \mathbb{R}\} \\ &\leq \sup\{|f(x)| \mid x \in \mathbb{R}\} + \sup\{|g(x)| \mid x \in \mathbb{R}\} = \|f\| + \|g\|. \end{aligned}$$

Example 8.2 *Let V be the space of sequences*

$$x = (\alpha_1, \alpha_2, \dots, \alpha_i, \dots)$$

of real numbers $\alpha_i \in \mathbb{R}$ in which at most finitely many $\alpha_i \neq 0$.
Define the operations 'addition' and 'multiplication with scalars' in V by the obvious coordinate-wise definitions. Furthermore, set

$$\|x\| = \sum_{i=1}^{\infty} |\alpha_i\|.$$

1. *Show that V is a vector space and that $\|\cdot\|$ is a norm in V.*
2. *Consider an infinite series of real numbers*

$$\sum_{i=1}^{\infty} a_i.$$

(There is no condition that at most finitely many $a_i \neq 0$.)
Define the sequence (x_n) in V by $x_1 = (a_1, 0, 0, \dots)$, $x_2 = (a_1, a_2, 0, \dots)$, and in general

$$x_n = (a_1, a_2, \dots, a_n, 0, \dots).$$

Show that the series $\sum_{i=1}^{\infty} |a_i|$ is convergent, if and only if the sequence (x_n) is a Cauchy sequence (fundamental sequence) in the normed vector space V, i.e.

$$\forall \varepsilon > 0 \exists n_0 \in \mathbb{N} \forall n, k \in \mathbb{N} : n \geq n_0 \quad \Longrightarrow \quad \|x_{n+k} - x_n\| < \varepsilon.$$

3. *Give an example of a Cauchy sequence in the normed vector space V that has no limit point in V.*

1. Let $x = (\alpha_1, \alpha_2, \dots, \alpha_i, \dots) \in V$ and $y = (\beta_1, \beta_2, \dots, \beta_i, \dots) \in V$, and $k \in \mathbb{R}$. Then

$$x + ky = (\alpha_1 + k\beta_1, \alpha_2 + k\beta_2, \dots, \alpha_i + k\beta_i, \dots),$$

and since x, $y \in V$, there exists an N, such that $\alpha_i = \beta_i = 0$ for every $i > N$. Then also $\alpha_i + k\beta_i = 0$ for every $i > N$, hence $x + ky \in V$, and V is a vector space.

It is obvious that $\|x\| \geq 0$, and if $\|x\| = \sum_{i=1}^{\infty} |\alpha_i| = 0$, then all $\alpha_i = 0$, thus $x = 0$.
Furthermore,

$$\|kx\| = \sum_{i=1}^{\infty} |k\alpha_i| = |k| \sum_{i=1}^{\infty} |\alpha_i| = |k| \cdot \|x\|,$$

and

$$\|x + y\| = \sum_{i=1}^{\infty} |\alpha_i + \beta_i\| \leq \sum_{i=1}^{\infty} |\alpha_i| + \sum_{i=1}^{\infty} |\beta_i| = \|x\| + \|y\|.$$

We have proved that $\|\cdot\|$ is a norm.

2. Assume that $\sum_{i=1}^{\infty} |a_i| < \infty$ is convergent. This means that

$$\forall \varepsilon > 0 \, \exists \, N : \sum_{i=N}^{\infty} |a_i| < \varepsilon.$$

Then for every $n \geq N$ and every $k \in \mathbb{N}$,

$$\|x_{n+k} - x_n\| = \sum_{i=n+1}^{n+k} |a_i| \leq \sum_{i=N}^{\infty} |a_i| < \varepsilon,$$

and (x_n) is a Cauchy sequence.

Conversely, if (x_n) is a Cauchy sequence,

$$\forall\, \varepsilon > 0\, \exists\, n_0 \in \mathbb{N}\, \forall\, n,\, k \in \mathbb{N} : n \geq n_0 \quad \Longrightarrow \quad \|x_{n+k} - x_n\| < \varepsilon,$$

then for $n = n_0$ and every $k \in \mathbb{N}$,

$$\sum_{i=n_0+1}^{n_0+k} |a_i| < \varepsilon, \quad \text{thus} \quad \lim_{k\to\infty} \sum_{i=n_0+1}^{n_0+k} |a_i| \leq \varepsilon,$$

or

$$\sum_{i=n_0+1}^{\infty} |a_i| \leq \varepsilon.$$

We conclude that

$$\left|\sum_{i=1}^{\infty}|a_i|-\sum_{i=1}^{n_0}|a_i|\right|=\sum_{i=n_0+1}^{\infty}|a_i|\le\varepsilon,$$

thus $\sum_{i=1}^{\infty}|a_i|<\infty$, and

$$\sum_{i=1}^{\infty}|a_i|=\sum_{k\in\mathbb{N}}\sum_{i=1}^{k}|a_i|\qquad\text{is uniquely determined.}$$

3. Choose $a_i=\dfrac{1}{i^2}$. Then

$$\sum_{i=1}^{\infty}\frac{1}{i^2}=\frac{\pi^2}{6}$$

is convergent, and (x_n) is a Cauchy sequence in V.
The limit point is

$$\left(1,\frac{1}{4},\frac{1}{9},\dots,\frac{1}{i^2},\dots\right)\notin V,$$

because every coordinate is >0.

Example 8.3 *Let V be a finite dimensional, normed vector space with norm $\|\cdot\|$. Let $L:V\to V$ be an arbitrary linear mapping. Show that there exists a unit vector $x_0\in V$, i.e. $\|x_0\|=1$, such that $\|L(x_0)\|=\|L\|$, where $\|L\|$ is the operator norm of L, i.e.*

$$\|L\|=\sup\{\|L(x)\|\mid\|x\|=1\}.$$

Show by an example that this does not hold in general, when V has infinite dimension.

Any finite dimensional and normed vector space V is isomorphic with $(\mathbb{R}^n,\|\cdot\|^\star)$ for some n and some norm $\|\cdot\|^\star$. In particular, the closed unit ball in V is compact.
Since $L:V\to V$ is continuous, there exists a point x_0 from the compact set $\{x\in V\mid\|x\|=1\}$, such that $\|x_0\|=1$, and such that

$$\|L\|=\sup\{\|L(x)\|\mid\|x\|=1\}=\|L(x_0)\|.$$

Then let V be the infinite dimensional vector space with consists of all summable sequences (x_n) of the norm

$$\|(x_n)\|_1=\sum_{n=1}^{\infty}|x_n|,$$

and let $L:V\to V$ denote the linear mapping which is degenerated by

$$L(e_n)=\left(1-\frac{1}{n}\right)e_n,\qquad n\in\mathbb{N}.$$

Clearly, $\|L\|\le 1$, and we conclude from

$$\lim_{n\to\infty}\|L(e_n)\|_1=\lim_{n\to\infty}\left(1-\frac{1}{n}\right)=1,$$

that $\|L\|=1$.
Finally, if $\|(x_n)\|_1=\sum_{n=1}^{\infty}|x_n|=1$ is any unit vector, then there exists an $n_0\in\mathbb{N}$, such that $|x_{n_0}|\neq 0$.
Then we have for every unit vector (x_n), $\|(x_n)\|_1=1$ that

$$\|L((x_n))\|_1=\sum_{n=1}^{\infty}\left(1-\frac{1}{n}\right)|x_n|<\sum_{n=1}^{\infty}|x_n|=1,$$

and $\|L((x_n))\|_1<1$ for every unit vector (x_n), so the claim does not hold in general for infinitely dimensional spaces.

9 Normed vector spaces and integral operators

Example 9.1 *Let $C([0,1],\mathbb{R})$ be the vector space of continuous real-valued functions in the unit interval $[0,1]$. For a continuous function $f:[0,1]\to\mathbb{R}$ we set*

$$\|f\|_1 = \int_0^1 |f(x)|\,dx.$$

1. *Show that $\|\cdot\|_1$ is a norm in $C([0,1],\mathbb{R})$.*

We now equip $C([0,1],\mathbb{R})$ as a normed vector space with the norm $\|\cdot\|_1$ and define the function

$$I : C([0,1],\mathbb{R}) \to \mathbb{R} \qquad \text{by} \qquad I(f) = \int_0^1 f(x)\,dx.$$

2. *Show that I is a continuous linear function.*

3. *Determine the operator norm of I.*

4. *In $C([0,1],\mathbb{R})$ equipped with the norm $\|\cdot\|_1$, consider the sequences (f_n) and (g_n) defined by*

$$f_n(x) = \begin{cases} 1-nx & \text{for } 0 \le x \le \dfrac{1}{n}, \\[2mm] 0 & \text{for } \dfrac{1}{n} \le x \le 1, \end{cases}$$

$$g_n(x) = \begin{cases} n-n^2x & \text{for } 0 \le x \le \dfrac{1}{n}, \\[2mm] 0 & \text{for } \dfrac{1}{n} \le x \le 1. \end{cases}$$

Examine the convergence of each of these sequences and, in case of convergence, determine the limit function.

1. Obviously, $\|f\|_1 \ge 0$. It follows from f being continuous and

$$\|f\|_1 = \int_0^1 |f(x)|\,dx = 0,$$

that $|f(x)| = 0$ for every x, so $f = 0$.

Remark 9.1 We give a simple indirect proof. Assume that $|f(x_0)| > 0$ for some $x_0 \in [0,1]$. Then there are a constant $c > 0$ and an interval J with $x_0 \in J$ of length $\varepsilon > 0$, such that $|f(x)| \ge c$ for every $x \in J$. Then we have the estimate

$$\|f\|_1 = \int_0^1 |f(x)|\,dx \ge \int_J |f(x)|\,dx \ge c\cdot\varepsilon > 0,$$

and the claim follows. This proof should be well-known to the reader, so it is only given here for completeness in a remark. $\Diamond$

Furthermore,

$$\|\alpha\cdot f\|_1 = \int_0^1 |\alpha\cdot f(x)|\,dx = |\alpha|\int_0^1 |f(x)|\,dx = |\alpha|\cdot\|f\|_1,$$

and

$$\|f+g\|_1 = \int_0^1 |f(x)+g(x)|\,dx \le \int_0^1 |f(x)|\,dx + \int_0^1 |g(x)|\,dx = \|f\|_1 + \|g\|_1,$$

and we have proved that $\|\cdot\|_1$ is a norm.

2. It follows from
$$|I(f)| = \left|\int_0^1 f(x)\,dx\right| \le \int_0^1 |f(x)|\,dx = \|f\|_1,$$
that I is continuous.
(To any $\varepsilon > 0$ choose $\delta = \varepsilon$, such that if $\|f\|_1 < \delta = \varepsilon$, then $|I(f)| \le \|f\|_1 < \varepsilon$).

3. It follows from the estimate in (2) that
$$\|I\| = \sup\{|I(f)| \mid \|f\|_1 = 1\} \le \sup\{\|f\|_1 \mid \|f\|_1 = 1\} = 1,$$
thus $\|I\| \le 1$.
On the other hand, if $f(x) \ge 0$, then
$$|I(f)| = \int_0^1 f(x)\,dx = \|f\|_1,$$
and $\|I\| \ge 1$.
We conclude that $\|I\| = 1$.

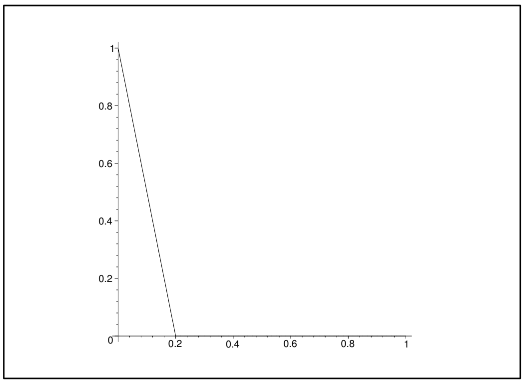


Figure 16: The graph of $f_5(x)$.

4. A simple figure shows that both (f_n) and (g_n) converge pointwise towards 0 for $x \in]0,1]$, so 0 is the only candidate of a limit value. We infer from
$$\|f_n - 0\|_1 = \|f_n\| = \frac{1}{2}\cdot 1 \cdot \frac{1}{n} = \frac{1}{2n} \to 0 \qquad \text{for } n \to \infty,$$
that $f_n \to 0$ for $n \to \infty$ in the norm $\|\cdot\|_1$.

This goes wrong for (g_n):
$$\|g_n - 0\|_1 = \|g_n\| = \frac{1}{2}\cdot n \cdot \frac{1}{n} = \frac{1}{2},$$
which does not converge towards the only possible limit value 0 for $n \to \infty$, and (g_n) is not convergent.

Example 9.2 *Let $f : E \to F$ be a mapping between normed vector spaces E and F which is differentiable at $0 \in E$ and has the property $f(\alpha h) = \alpha f(h)$ for all $\alpha \in \mathbb{R}$ and all $h \in E$. Show that f is linear.*

When f is differentiable at $0 \in E$, then there exists a linear mapping $Df(0) : E \to F$, such that
$$f(x) = f(x) - f(0) = Df(0)x + \varepsilon(x)\|x\|_E \qquad \text{for alle } x \in E,$$

where we have used that $f(0) = f(0 \cdot 0) = 0 \cdot f(0) = 0$.

Insert $\alpha(x+y)$ instead of x. Then

$$\begin{aligned}\alpha f(x+y) &= f(\alpha x + \alpha y) = Df(0)(\alpha x + \alpha y) + \varepsilon(\alpha(x+y))\|\alpha(x+y)\|_E \\ &= \alpha Df(0)x + \alpha Df(0)y + \varepsilon(\alpha(x+y)) \cdot |\alpha|\, \|x+y\|_E .\end{aligned}$$

When this identity is divided by $\alpha \neq 0$, we get with another ε-function,

$$f(x+y) = Df(0)x + Df(0)y + \varepsilon(\alpha(x+y)) \cdot \|x+y\|_E .$$

It follows by taking the limit $\alpha \to 0$ that

$$f(x+y) = Df(0)x + Df(0)y.$$

An analogous, though simpler argument shows that

$$f(x) = Df(0)x \qquad \text{and} \qquad f(y) = Df(0)y,$$

thus

$$f(x+y) = Df(0)x + Df(0)y = f(x) + f(y).$$

Finally,

$$f(x+\lambda y) = f(x) + f(\lambda y) = f(x) + \lambda f(y).$$

This holds for every x, $y \in E$ and every $\lambda \in \mathbb{R}$, so we have proved that f is linear.

Example 9.3 *Let $C([0,1],\mathbb{R})$ be the vector space of continuous functions $f : [0,1] \to \mathbb{R}$ equipped as a normed vector space with the norm*

$$\|f\| = \sup\{|f(x)| \mid 0 \le x \le 1\}.$$

Let $\Phi = \Phi(x,y) : [0,1]\times[0,1] \to \mathbb{R}$ be a continuous function in two variables defined in the square $[0,1]\times[0,1]$ in $\mathbb{R}^2$. Assume that $\Phi(x,y) \ge 0$ for all $(x,y) \in [0,1]\times[0,1]$.
Define the function $\varphi = \varphi(x) : [0,1] \to \mathbb{R}$ by

$$\varphi(x) = \sup\{\Phi(x,y) \mid 0 \le y \le 1\}.$$

For $f \in C([0,1],\mathbb{R})$ we define the function $f_\Phi = f_\Phi(y) : [0,1] \to \mathbb{R}$ by

$$f_\Phi(y) = \int_0^1 \Phi(x,y) f(x)\, dx.$$

1. *Show that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that*

- **(a)** $\Phi(x_0,y) - \varepsilon \le \Phi(x,y) \le \Phi(x_0,y) + \varepsilon$ *for $|x - x_0| \le \delta$ and all $y \in [0,1]$.*
- **(b)** $|\Phi(x,y) - \Phi(x,y_0)| \le \varepsilon$ *for $|y - y_0| \le \delta$ and all $x \in [0,1]$.*

Make use of this to show that the functions $\varphi = \varphi(x)$ and $f_\Phi = f_\Phi(y)$ are continuous.

Since $f_\Phi \in C([0,1]\times\mathbb{R})$, we can define the mapping

$$L : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R}) \qquad by \qquad L(f) = f_\Phi.$$

The mapping L is called an integral operator *with kernel Φ.*

2. *Show that L is a continuous linear mapping.*

3. *Show that*

$$\|L(f)\| \le \left(\int_0^1 \varphi(x)\, dx\right) \|f\|$$

for all $f \in C([0,1],\mathbb{R})$.

4. *Show that the operator norm for L is given by*

$$\|L\| = \sup\left\{ \int_0^1 \Phi(x,y)\, dx \;\middle|\; 0 \le y \le 1 \right\}.$$

1. The mapping Φ is continuous on the compact set $[0,1]\times[0,1]$, hence Φ is uniformly continuous, thus to every $\varepsilon > 0$ there exists a $\delta > 0$, such that if (x_0,y_0) and $(x,y) \in [0,1]\times[0,1]$ satisfy the conditions

   $$|x - x_0| < \delta \qquad \text{and} \qquad |y - y_0| < \delta,$$

   then

   $$|\Phi(x_0,y_0) - \Phi(x,y)| < \varepsilon.$$

   If in particular, $y = y_0$, then of course $|y - y_0| = 0 < \delta$, and it follows that

   $$\text{if } |x - x_0| < \delta, \text{ then } |\Phi(x_0,y) - \Phi(x,y)| < \varepsilon,$$

which is also written

$$\Phi(x_0,y)-\varepsilon \le \Phi(x,y) \le \Phi(x_0,y)+\varepsilon \quad \text{for } |x-x_0|\le\delta,\ y\in[0,1].$$

When we repeat the argument with $|y-y_0|<\delta$ and $x=x_0\in[0,1]$, we get

$$|\Phi(x,y)-\Phi(x,y_0)|\le\varepsilon \qquad \text{for } |y-y_0|\le\delta \text{ and } x\in[0,1].$$

It follows from the estimates

$$\Phi(x_0,y)-\varepsilon \le \Phi(x,y) \le \Phi(x_0,y)+\varepsilon$$

for $|x-x_0|\le\delta$ and $y\in[0,1]$ that

$$\begin{aligned}\sup\{\Phi(x_0,y)\mid y\in[0,1]\}-\varepsilon &\le \sup\{\Phi(x,y)\mid y\in[0,1]\}\\ &\le \sup\{\Phi(x_0,y)\mid y\in[0,1]\}+\varepsilon,\end{aligned}$$

and then we use the definition of φ to imply that

$$\varphi(x_0)-\varepsilon\le\varphi(x)\le\varphi(x_0)+\varepsilon,$$

thus $|\varphi(x)-\varphi(x_0)|<\varepsilon$, which holds whenever $|x-x_0|<\delta$. This proves that φ is continuous.

If $|y-y_0|<\delta$, then we get the estimates

$$\begin{aligned}|f_\Phi(y)-f_\Phi(y_0)| &= \left|\int_0^1\{\Phi(x,y)-\Phi(x,y_0)\}\,f(x)\,dx\right|\\ &\le \int_0^1|\Phi(x,y)-\Phi(x,y_0)|\cdot|f(x)|\,dx\\ &\le \varepsilon\int_0^1|f(x)|\,dx=\|f\|_1\cdot\varepsilon.\end{aligned}$$

This implies that$f_\Phi:[0,1]\to\mathbb{R}$ is continuous.

2. Clearly,

$$\begin{aligned}L(f+\lambda g) &= \int_0^1\Phi(x,y)\{f(x)+\lambda g(x)\}\,dx\\ &= \int_0^1\Phi(x,y)f(x)\,dx+\lambda\int_0^1\Phi(x,y)g(x)\,dx\\ &= L(f)(y)+\lambda L(g)(y),\end{aligned}$$

hence $L(f+\lambda g)=L(f)+\lambda L(g)$, and the mapping L is linear.
Furthermore, $|\Phi(x,y)|$ has a maximum, $\|\Phi\|_\infty$ på $[0,1]\times[0,1]$,

$$\|\Phi\|_\infty=\sup\{|\Phi(x,y)|\mid(x,y)\in[0,1]\times[0,1]\},$$

because Φ is continuous on the compact set $[0,1]\times[0,1]$. Then

$$\begin{aligned}\|L(f)\| &= \sup\{|L(f)(y)|\mid 0\le y\le 1\}=\sup\left\{\left|\int_0^1\Phi(x,y)f(x)\,dx\right|\;\middle|\;0\le y\le 1\right\}\\ &\le \sup\left\{\int_0^1|\Phi(x,y)|\cdot|f(x)|\,dx\;\middle|\;0\le y\le 1\right\}\\ &\le \sup\left\{\int_0^1\|\Phi\|_\infty\cdot\|f\ \ dx\;\middle|\;0\le y\le 1\right\}=\|\Phi\|_\infty\cdot\|f\|.\end{aligned}$$

If $\Phi=0$, then $L(f)=0$ which is trivially continuous.
If $\Phi\ne0$, then $\|\Phi\|_\infty>0$. Choose to $\varepsilon>0$, the δ by

$$\delta=\frac{\varepsilon}{\|\Phi\|_\infty}.$$

If $\|f\|<\delta$, then $\|L(f)\|<\varepsilon$, proving that L is continuous at 0. Since L is linear and continuous at 0, it is continuous everywhere.

3. In this case we use the estimate above in the following way:

$$\begin{aligned}\|L(f)\| &\le \sup\left\{\int_0^1 |\Phi(x,y)|\cdot|f(x)|\,dx \;\middle|\; 0\le y\le 1\right\} \le \int_0^1 \sup\{|\Phi(x,y)| \mid 0\le y\le 1\}\cdot\|f\|\,dx \\ &= \left(\int_0^1 \varphi(x)\,dx\right)\|f\|.\end{aligned}$$

4. From $\Phi(x,y)\ge 0$, and

$$\begin{aligned}\|L(1)\| &= \sup\left\{\int_0^1 \Phi(x,y)\,dx \;\middle|\; 0\le y\le 1\right\} \le \|L\| \\ &= \sup_{\|f\|=1}\sup\left\{\int_0^1 \Phi(x,y)f(x)\,dx \;\middle|\; 0\le y\le 1\right\} \\ &\le \sup\left\{\int_0^1 \Phi(x,y)\,dx \;\middle|\; 0\le y\le 1\right\},\end{aligned}$$

(because $|f(x)|\le 1$), follows that

$$\|L\| = \sup\left\{\int_0^1 \Phi(x,y)\,dx \;\middle|\; 0\le y\le 2\right\}.$$

10 Differentiable mappings

Example 10.1 *Let U be an open set in a normed vector space E. A real-valued function $f : U \to \mathbb{R}$ is said to have a* local maximum (minimum) *at a point $x_0 \in U$ if there exists a neighbourhood $N \subseteq U$ of x_0 such that $f(x) \le f(x_0)$ $[f(x) \ge f(x_0)]$ for all $x \in N$.*

1. *Suppose that the function $f : U \to \mathbb{R}$ is differentiable at the point $x_0 \in U$. Prove that for each fixed $h \in E$, there exists an $r > 0$ such that the function $g(t) = f(x_0 + th)$ is defined for $t \in]-r, r[$ and is differentiable at 0 with derivative $g'(0) = Df(x_0)(h)$.*

2. *Suppose that the function $f : U \to \mathbb{R}$ is differentiable at the point $x_0 \in U$ and that f has the local maximum (minimum) at $x_0 \in U$. Prove that the differential of f at x_0 is zero, i.e. $Df(x_0) = 0$.*

1. When $f : U \to \mathbb{R}$ is differentiable at $x_0 \in U$, then

$$f(x_0 + h) - f(x_0) = Df(x_0)(h) + \varepsilon(h)\|h\|,$$

hence

$$g(t) = f(x_0 + th) = f(x_0) + t \cdot Df(x_0)(h) + \varepsilon(th) \cdot t\|h\|.$$

Then $g(0) = f(x_0)$, and

$$g(t) - g(0) = t \cdot Df(x_0)(h) + t \cdot \varepsilon(th)\|h\|,$$

hence

$$\lim_{t\to 0} \frac{g(t) - g(0)}{t - 0} = Df(x_0)(h) + \lim_{t\to 0} \varepsilon(th) \cdot \|h\| = Df(x_0)(h),$$

i.e. $g'(0) = Df(x_0)(h)$.

2. Assume e.g. that $f(x) \le f(x_0)$ for every $x \in N$. (Analogously, if $f(x) \ge f(x_0)$). Then

$$0 \ge f(x_0 + h) - f(x_0) = Df(x_0)(h) + \varepsilon(h)\|h\|,$$

hence

$$\lim_{h\to 0} Df(x_0)\left(\frac{h}{\|h\|}\right) \le 0$$

and

$$\lim_{h\to 0} Df(x_0)\left(-\frac{h}{\|h\|}\right) = -\lim_{h\to 0} Df(x_0)\left(\frac{h}{\|h\|}\right) \le 0,$$

so

$$\lim_{h\to 0} Df(x_0)\left(\frac{h}{\|h\|}\right) = 0.$$

We get that $Df(x_0) = 0$, because $h/\|h\|$ is an arbitrary unit vector.

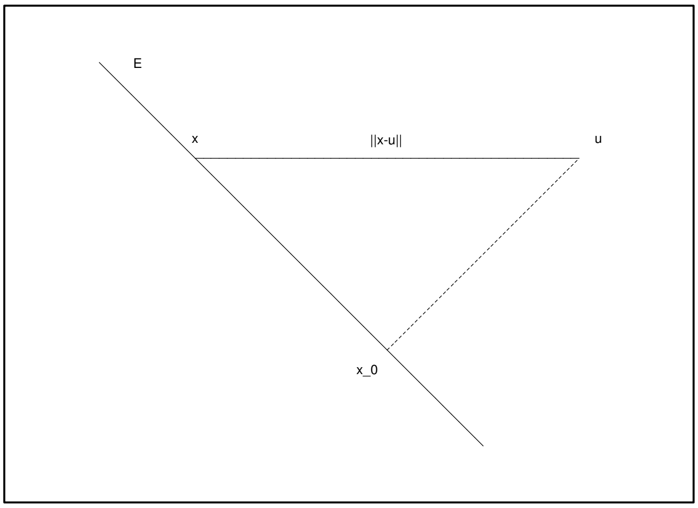
Example 10.2 *Let $\mathcal{H}$ denote a vector space with inner product $\langle \cdot , \cdot \rangle$ and associated norm $\| \cdot \|$ defined by $\|x\| = \sqrt{\langle x, x\rangle}$ for $x \in \mathcal{H}$.* (Example: $\mathcal{H} = \mathbb{R}^n$ equipped with the standard inner product $\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$.)
Let E denote a finite dimensional proper subspace of $\mathcal{H}$ and let $u \in \mathcal{H}$ be a fixed point in $\mathcal{H}$ outside E.
Define the function $f : E \to \mathbb{R}$ by

$$f(x) = \|x - u\|^2 = \langle x - u, x - u\rangle \qquad \text{for } x \in E.$$

1. *Prove that f is differentiable at every point $x \in E$ with differential $Df(x) : E \to \mathbb{R}$ given by*

$$Df(x)(h) = 2\langle x - u, h\rangle \qquad \text{for all } h \in E.$$

2. *Prove that the differential of f is zero at exactly one point $x_0 \in E$.*
HINT: The differential of f is zero at $x_0 \in E$, i.e. $Df(x_0) = 0$, if and only if the vector $x_0 - u$ is orthogonal to E.



1. We shall only rearrange a little for x, $h \in E$,

$$\begin{aligned} f(x+h) - f(x) &= \langle x + h - u, x + h - u\rangle - \langle x - u, x - u\rangle \\ &= \langle (x-u) + h, (x-u) + h\rangle - \langle x - u, x - u\rangle \\ &= \{\langle x - u, x - u\rangle + 2\langle x - u, h\rangle + \langle h, h\rangle\} - \langle x - u, x - u\rangle \\ &= 2\langle x - u, h\rangle + \|h\|^2, \end{aligned}$$

hence

$$Df(x)(h) = 2\langle x - u, h\rangle \quad \text{og} \quad \|h\|^2 = \varepsilon(h) \cdot \|h\|, \quad \text{med} \quad \varepsilon(h) = \|h\|,$$

and the claim is proved.

2. Let $x_0 \in E$ be the point of the minimum for $f(x) = d(x, u)^2$. It exists, because $E \cap \overline{B}_R(u)$ is compact for $R > \operatorname{dist}(E, u)$, and $f(x)$, $x \in E$, is continuous. It follows from EXAMPLE 10.1 that $Df(x_0) = 0$, so $x_0 - u$ is perpendicular to E.

Any other $x \in E$ can be written $x = x_0 + h$, $h \in E$. From $h \perp x_0 - u$, and Pythagoras's theorem follow that

$$\|x - u\|^2 = \|x_0 - u\|^2 + \|h\|^2,$$

and

$$2\langle x-u,k\rangle = 2\langle x_0-u+h,k\rangle = 2\langle h,k\rangle, \qquad k\in E, \quad h\in E.$$

Choosing $k=h\neq 0$ we see that the differential is $\neq 0$ at $x=x_0+h$, $h\in E\setminus\{0\}$, and the claim is proved.

Example 10.3 *Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space equipped with the usual inner product*

$$\langle x,y\rangle = \sum_{i=1}^n x_i y_i,$$

and the associated norm

$$\|x\| = \sqrt{\langle x,x\rangle} = \sqrt{\sum_{i=1}^n x_i x_i}.$$

Denote by $E=C^1([a,b],\mathbb{R}^n)$ the space of differentiable real-valued functions $f:[a,b]\to\mathbb{R}^n$ of class C^1 defined on the interval $[a,b]$ in $\mathbb{R}$. We can equip E with the structure of a normed vector space with norm

$$\|f\|_1 = \sup_{a\le t\le b}(\|f(t)\|+\|f'(t)\|).$$

Define the (kinetic) energy function $K:E\to E$ by

$$K(f) = \frac{1}{2}\int_a^b \|f'(t)\|^2 dt = \frac{1}{2}\int_a^b \langle f'(t),f'(t)\rangle\,dt \qquad \text{for } f\in E.$$

1. *Prove that K is differentiable at every $f\in E$ with differential $DK(f):E\to\mathbb{R}$ given by*

$$DK(f)(h) = \int_a^b \langle f'(t),h'(t)\rangle\,dt \qquad \text{for all } h\in E.$$

2. *Prove that the differential of K at $f\in E$ is zero, i.e. $DK(f)=0$, if and only if f is a constant function.* Hint: (Try to set $h=f$.)

3. *Provide a physical interpretation of the result in (2).*

1. By a computation,

$$\begin{aligned} &K(f+h)-K(f)\\ &= \frac{1}{2}\int_a^b \langle f'(t)+h'(t),f'(t)+h'(t)\rangle\,dt - \frac{1}{2}\int_a^b \langle f'(t),f'(t)\rangle\,dt\\ &= \frac{1}{2}\int_a^b \{\langle f'(t),f'(t)\rangle + 2\langle f'(t),h'(t)\rangle + \langle h'(t),h'(t)\rangle - \langle f'(t),f'(t)\rangle\}\,dt\\ &= \int_a^b \langle f'(t),h'(t)\rangle\,dt + \frac{1}{2}\int_a^b \|h'(t)\|^2 dt, \end{aligned}$$

where

$$\frac{1}{2}\int_a^b \|h'(t)\|^2 dt \le \frac{1}{2}(b-a)\cdot\|h\|_1^2 = \frac{1}{2}(b-a)\|h\|_1\cdot\|h\|_1,$$

thus

$$0\le\varepsilon(h)\le\frac{1}{2}(b-a)\|h\|_1\to 0 \qquad \text{for } \|h\|_1\to 0.$$

The first term is linear in h for fixed f, thus

$$DK(f)(h) = \int_a^b \langle f'(t),h'(t)\rangle\,dt \qquad \text{for every } f\in E, \quad h\in E.$$

2. If f is constant, then clearly $f'(t) = 0$, hence

$$DK(f)(h) = \int_a^b \langle 0, h'(t)\rangle\, dt = 0 \qquad \text{for alle } h \in E,$$

så $DK(f) = 0$.

If f is not constant, then $f' \neq 0$. Choosing $h = f$ we get

$$DK(f)(f) = \int_a^b \langle f'(t), f'(t)\rangle\, dt = \int_a^b \|f'(t)\|^2 dt > 0,$$

which shows that $DK(f) \neq 0$, and the claim is proved.

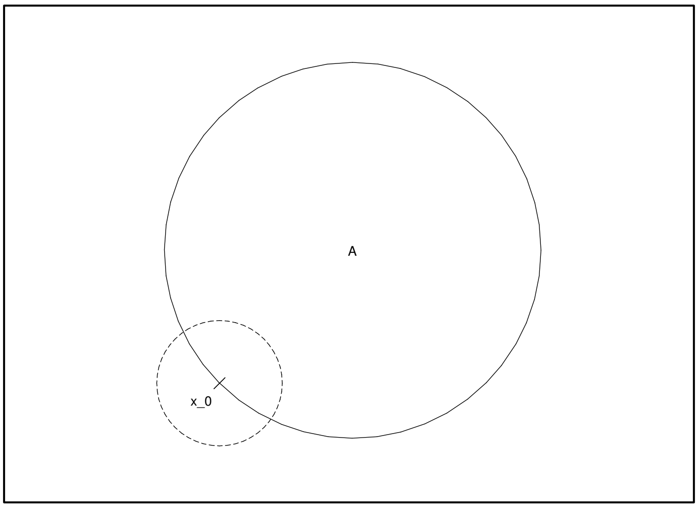
3. Let $f(t)$ denote the space coordinate of a particle, which moves along the X-axis. The velocity is $f'(t)$, and the kinetic energy is

$$K(f) = \frac{1}{2}\int_a^b \|f'(t)\|^2 dt.$$

According to (2) the differential is $DK(f) = 0$, if and only if $f(t)$ is constant, i.e. if and only if the particle is at rest.

11 Complete metric spaces

Example 11.1 *Let (S, d) be a metric space. Show that if a subset A in S is a complete metric space in the induced metric from S, then A is a closed set in S.*



Indirect proof. Let $x_0 \in \overline{A} \setminus A$, i.e.

$$A \cap B_r(x_0) \neq \emptyset \qquad \text{for every } r > 0.$$

To a given $r = \dfrac{1}{n}$ we choose $x_n \in A \cap B_{1/n}(x_0)$. Then $x_n \to x_0$ in S for $n \to \infty$. A convergent sequence is also a Cauchy sequence, thus (x_n) is a Cauchy sequence in both S and A.

In a metric space a possible limit point for a Cauchy sequence is always unique. The limit point is $x_0 \notin A$, and we have constructed a Cauchy sequence on A, which is not convergent in A. Hence, A is not complete in the induced topology.

We get by contraposition that if A is complete in the induced topology, then A is closed.

Example 11.2 *Let X be a compact topological space, and let S be a complete metric space with metric d.*
By $C(X,S)$ we denote the space of continuous mappings $f : X \to S$.
For f, $g \in C(X,S)$ we put

$$D(f,g) = \sup_{x\in X} d(f(x), g(x)).$$

Then D is a metric in $C(X,S)$.

1. *Show that (f_n) is a Cauchy sequence in the metric space $(C(X,S), D)$ if and only if $(f_n : X \to S)$ is a uniform Cauchy sequence, i.e.*

$$\forall \varepsilon > 0 \exists n_0 \in \mathbb{N} \forall n,\, m \in \mathbb{N} : n,\, m \geq n_0 \quad \Longrightarrow \quad \forall x \in X : d(f_n(x), f_m(x)) < \varepsilon.$$

Now let (f_n) be a Cauchy sequence in $(C(X,S), D)$.

2. *Show that for every $x \in X$, there exists a uniquely determined $y \in S$, such that $f_n(x) \to y$ for $n \to \infty$.*

Define a mapping $f : X \to S$ by setting $f(x) = y$ for all $x \in X$, where $y \in S$ is determined as in (2). In other words, the mapping f is defined by

$$f(x) = \lim_{n\to\infty} f_n(y) \qquad \text{for } x \in X.$$

3. *First show that (f_n) converges uniformly to f for n going to ∞, i.e.*

$$\forall \varepsilon > 0 \exists n_0 \in \mathbb{N} \forall n \in \mathbb{N} : n \geq n_0 \quad \Longrightarrow \quad \forall x \in X : d(f_n(x), f(x)) < \varepsilon.$$

Next show that $f : X \to S$ is continuous.
HINT:

$$d(f(x), f(x_0)) \leq d(f(x), f_n(x)) + d(f_n(x), f_n(x_0)) + d(f_n(x_0), f(x_0)).$$

4. *Show that $(C(X,S), D)$ is a complete metric space.*

1. Since d is a metric, it follows that

$$D(f,g) = \sup_{x\in X} d(f(x), g(x)) \geq 0\}.$$

Furthermore, since X is *compact*, we have $D(f,g) < \infty$, so D is defined.
If $D(f,g) = 0$, then $d(f(x), g(x)) = 0$ for every $x \in X$, thus $f = g$.
Furthermore,

$$D(f,g) = \sup_{x\in X} d(f(x), g(x)) = \sup_{x\in X} d(g(x), f(x)) = D(g,f),$$

and

$$\begin{aligned} D(f,g) &= \sup_{x\in X} d(f(x), g(x)) \\ &\leq \sup_{x\in X} \{d(f(x), h(x)) + d(h(x), g(x))\} \\ &\leq \sup_{x\in X} d(f(x), h(x)) + \sup_{x\in X} d(h(x), g(x)) \\ &= D(f,h) + D(h,g), \end{aligned}$$

and we have proved that D is a metric on $C(X,S)$.

Assume that (f_n) is a Cauchy sequence in $(C(X,S), D)$. Then

(3) $\quad \forall \varepsilon > 0 \exists n_0 \in \mathbb{N} \forall m,\, n \geq n_0 : D(f_m, f_n) < \varepsilon.$

Now,

$$D(f_m, f_n) = \sup_{x \in X} d(f_m(x), f_n(x)) \geq d(f_m(x), f_n(x))$$

for all $x \in X$, so

(4) $\quad \forall \varepsilon > 0 \exists n_0 \in \mathbb{N} \forall m, n \geq n_0 \forall x \in X : d(f_m(x), f_n(x)) < \varepsilon.$

We see that this is the condition that (f_n) is a uniform Cauchy sequence.

Conversely, if (f_n) is a uniform Cauchy sequence, then (4) holds, and thus in particular

$$d(f_m(x), f_n(x)) < \varepsilon \qquad \text{for alle } x \in X,$$

and it follows that

$$D(f_m, f_n) = \sup_{x \in X} d(f_m(x), f_n(x)) \leq \varepsilon.$$

The only difference from the above is that we here have "$\leq \varepsilon$" instead of "$< \varepsilon$", so we derive again (3). This means that (f_n) is a Cauchy sequence in $(C(X,S), D)$, and (1) is proved.

2. Then let (f_n) be a Cauchy sequence in $(C(X,S),D)$. It follows from (1) that (f_n) is a uniform Cauchy sequence. In particular, $(f_n(x))$ is a Cauchy sequence on S for every $x \in X$. Since S is complete, $(f_n(x))$ is convergent for every $x \in X$, hence

$$\forall x \in X \exists y \in S : \lim_{n\to\infty} f_n(x) = y.$$

Now, y corresponds uniquely to x, so

$$f(x) = \lim_{n\to\infty} f_n(x), \qquad x \in X,$$

is a well-defined mapping.

3. The sequence (f_n) is a uniform Cauchy sequence, and the pointwise limit function f exists everywhere. Hence, (f_n) is uniformly convergent with the limit function f.

We shall prove that the limit function f is continuous. Using the hint we consider the estimate

$$(5) \qquad d(f(x),f(x_0)) \leq d(f(x),f_n(x)) + d(f_n(x),f_n(x_0)) + d(f_n(x_0),f(x_0)).$$

It follows from (4) that given any $\varepsilon > 0$ we can find an n_0, such that

$$d(f_m(x),f_n(x)) < \frac{\varepsilon}{6} \qquad \text{for every } x \in X,\ \text{if } m,\, n \geq n_0.$$

It follows that

$$d(f(x),f_n(x)) < \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{3} \quad \text{for every } x \in X \text{ and } n \geq n_0,$$

and we now control the first and the third term of (5).
Using that $f_n(x)$ is uniformly continuous there is a $\delta > 0$, such that

$$d(f_n(x),f_n(x_0)) < \frac{\varepsilon}{3}, \qquad \text{if } x \in U_\varepsilon(x_0),$$

where $U_\varepsilon(x_0)$ is an open neighbourhood corresponding to ε and x_0.

Hence, if $x \in U_\varepsilon(x_0)$, then

$$d(f(x),f(x_0)) \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,$$

and f is continuous at every $x_0 \in X$, i.e. in all of X.

4. We shall only collect all the previous results: If (f_n) is a Cauchy sequence in $(C(X,S),D)$, then (f_n), $f_n : X \to S$, is a uniform Cauchy sequence of limit function f, where this limit function $f \in C(X,S)$ is also continuous.

In other words: The Cauchy sequence (f_n) converges uniformly (hence also in the metric D) towards a continuous function $f \in C(X,S)$, and $(C(X,S),D)$ is a complete space.

12 Local Existence and Uniqueness Theorem for Autonomous Ordinary Differential Equations

Example 12.1 *(Local Existence and Uniqueness Theorem for Autonomous Ordinary Differential Equations.)*
Let E be a Banach space, and let $U \subseteq E$ be an open set in E. The norm in E is denoted by $\| \cdot \|$.
A mapping $f : U \to E$ is said to be Lipschitz continuous *in U, if there exists a constant k, such that*

$$\|f(x_1) - f(x_2)\| \leq k\|x_1 - x_2\| \qquad \text{for all } x_1,\, x_2 \in U.$$

Now assume that $f : U \to E$ is a Lipschitz continuous mapping in U. (One can think of f as a vector field in U by placing the vector $f(x) \in E$ at every point $x \in U$.)
Consider the Initial Value Problem consisting of the differential equation (i) below together with the initial value condition (ii):

$$(i)\;\; \frac{dx}{dt} = f(x), \qquad\qquad (ii)\;\; x(0) = x_0 \in U.$$

By a solution*, or, an* integral curve*, to the Initial Value Problem (i) and (ii) we understand a differential curve $\varphi : J \to U$ defined in an interval J around $0 \in \mathbb{R}$, such that*

$$\frac{d\varphi}{dt} = f(\varphi(t)) \qquad \text{for all } t \in J,$$

and such that $\varphi(0) = x_0$.

1. *Show that $\varphi : J \to U$ solves (i) and (ii) if and only if φ satisfies the integral equation*

$$\varphi(t) = x_0 + \int_0^t f(\varphi(\tau))\, d\tau.$$

For $a > 0$, let J_a denote the interval $J_a = [-a, a]$, and for $b > 0$, let

$$S_b = \{x \in E \mid \|x - x_0\| \leq b\}$$

denote the closed ball in E with centre $x_0 \in U$ and radius b. For $b > 0$ sufficiently small, we have $S_b \subseteq U$, and we shall only consider such b.
Let $C(J_a, S_b)$ denote the space of continuous mappings $\varphi : J_a \to S_b$ equipped with the metric D as in Example 11.2.

To $\varphi \in C(J_a, S_b)$ we associate $\psi : J_a \to E$ defined by

$$\psi(t) = x_0 + \int_0^t f(\varphi(\tau))\, d\tau \qquad \text{for } t \in J_a.$$

2. *Show that for sufficiently small $a > 0$, the mapping $\psi \in C(J_a, S_b)$.*

3. *Show that for sufficiently small $a > 0$, the mapping*

$$T : C(J_a, S_b) \to C(J_a, S_b),$$

which assigns $\psi = T(\varphi) \in C(J_a, S_b)$ to $\varphi \in C(J_a, S_b)$, is a contraction.

4. *Show that with $a > 0$ as in (3), there exists a unique solution $\varphi \in C(J_a, S_b)$ to the differential equation*

$$\frac{dx}{dt} = f(x),$$

such that $\varphi(0) = x_0$.

5. *Show that if $\varphi_1 : J_1 \to U$ and $\varphi_2 : J_2 \to U$ are two solutions to the differential equation*

$$\frac{dx}{dt} = f(x),$$

defined in overlapping open intervals J_1 and J_2 in $\mathbb{R}$, such that $\varphi_1(t_0) = \varphi_2(t_0)$ at a point $t_0 \in J_1 \cap J_2$, then $\varphi_1(t) = \varphi_2(t)$ at all points $t \in J_1 \cap J_2$.

6. *Show that there exists a unique maximal solution to the Initial Value Problem (i) and (ii).* (A maximal solution in a solution with an open interval of definition that cannot be extended.)

1. It follows from $\dfrac{d\varphi}{dt} = f(\varphi(t))$ by an integration that

$$[\varphi(\tau)]_0^t = \varphi(t) - \varphi(0) = \int_0^t f(\varphi(\tau))\, d\tau,$$

so we get since $\varphi(0) = x_0$ that

$$\varphi(t) = x_0 + \int_0^t f(\varphi(\tau))\, d\tau.$$

Conversely, if φ is given by this integral equation, then $\varphi(0) = x_0 + 0 = x_0$, and

$$\frac{d\varphi}{dt} = [f(\varphi(\tau))]_{\tau=t} = f(\varphi(t)),$$

and the claim is proved.

2. From $S_b \subseteq U$ follows that $\varphi(t) \in U$ for every $t \in J_a$, hence

$$\begin{aligned}\psi(t)-\psi(t_0) &= \left\{x_0+\int_0^t f(\varphi(\tau))\,d\tau\right\}-\left\{x_0+\int_0^{t_0} f(\varphi(\tau))\,d\tau\right\}\\ &= \int_{t_0}^t f(\varphi(\tau))\,d\tau=\int_{t_0}^t \{f(\varphi(\tau))-f(\varphi(t_0))\}\,d\tau+\int_{t_0}^t f(\varphi(t_0))\,d\tau,\end{aligned}$$

and we get the estimate

$$\begin{aligned}\|\psi(t)-\psi(t_0)\| &\le \left|\int_{t_0}^t \|f(\varphi(\tau))-f(\varphi(t_0))\|\,d\tau\right|+\|f(\varphi(t_0)\|\cdot|t-t_0|\\ &\le k\left|\int_{t_0}^t \|\varphi(\tau)-\varphi(t_0)\|\,d\tau\right|+\|f(\varphi(t_0))\|\cdot|t-t_0|.\end{aligned}$$

Since $\|f(\varphi(t_0))\|$ is a fixed number, we can choose $\delta_1>0$, such that

$$\|f(\varphi(t_0))\|\cdot|t-t_0|<\frac{\varepsilon}{2} \qquad \text{for } |t-t_0|<\delta_1.$$

Now, φ is continuous, so to every $\varepsilon_1>0$ we can choose $\delta_2>0$, such that

$$\|\varphi(\tau)-\varphi(t_0)\|<\varepsilon_1 \quad \text{for} \quad |\tau-t_0|<\delta_2.$$

In this case we have

$$k\left|\int_{t_0}^t \|\varphi(\tau)-\varphi(t_0)\|\,d\tau\right|\le k\cdot\varepsilon_1\cdot\delta_2<\frac{\varepsilon}{2}$$

for $\varepsilon_1,\ \delta_2>0$ sufficiently small. (It suffices to choose $\varepsilon_1>0$, and then $\delta_3=\min\left\{\delta_2,\frac{\varepsilon}{2k}\right\}$). Then

$$\|\psi(t)-\psi(t_0)\|\le\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

for $|t-t_0|<\delta=\min\{\delta_1,\delta_3\}$, thus $\psi\in C(J_a,S_b)$.

3. We have

$$T(\varphi_1)(t)-T(\varphi_2)(t)=\left\{x_0+\int_0^t f(\varphi_1(\tau))\,d\tau\right\}-\left\{x_0+\int_0^t f(\varphi_2(\tau))\,d\tau\right\}=\int_0^t\{f(\varphi_1(\tau))-f(\varphi_2(\tau))\}\,d\tau,$$

so by the Lipschitz condition,

$$\|T(\varphi_1)(t)-T(\varphi_2)(t)\|\le\left|\int_0^t\|f(\varphi_1(\tau))-f(\varphi_2(\tau))\|\,d\tau\right|\le k\cdot\left|\int_0^t\|\varphi_1(\tau)-\varphi_2(\tau)\|\,d\tau\right|.$$

If D_a is the metric on $C(J_a,S_b)$, given by

$$D_a(\varphi_1,\varphi_2)=\sup_{t\in J_a}\|\varphi_1(t)-\varphi_2(t)\|,$$

then it follows that

$$D_a(T(\varphi_1),T(\varphi_2))\le k\cdot\left|\int_0^t D_a(\varphi_1,\varphi_2)\,d\tau\right|\le k\cdot a\cdot D_a(\varphi_1,\varphi_2).$$

Choose $a>0$, such that $k\cdot a<1$. Then $T:C(J_a,S_b)\to C(J_a,S_b)$ is a contraction.

4. It follows from Example 11.2 that $(C(J_a,S_b),D_a)$ is a complete metric space. By Banach's fixpoint theorem the contraction T has only one fixpoint, thus there exists a unique $\varphi\in C(J_a,S_b)$, such that $\varphi(t)=T(\varphi)(t)$. This means that

$$\varphi(t)=x_0+\int_0^t f(\varphi(\tau))\,d\tau.$$

This is by (1) equivalent with that $x=\varphi(t)$ is the unique solution of

$$\frac{dx}{dt}=f(x), \qquad x(0)=x_0.$$

5. Let $\varphi_1 : J_1 \to U$ and $\varphi_2 : J_2 \to U$ be two solutions which agree in a point $t_0 \in J_1 \cap J_2$, where both J_1 and J_2 are closed intervals. We claim that

$$\varphi_1(t) = \varphi_2(t) \qquad \text{for every } t \in J_1 \cap J_2.$$

The mapping $\varphi_1 - \varphi_2$ is continuous, so

$$\{t \in J_1 \cap J_2 \mid \varphi_1(t) = \varphi_2(t)\} = (\varphi_1 - \varphi_2)^{\circ -1}(\{0\}) \cap (J_1 \cap J_2)$$

is a closed and nonempty set. If it is not all of $J_1 \cap J_2$, then the set $(\varphi_1 - \varphi_2)^{\circ -1}(\{0\})$ must have a boundary point t_1, which lies in the interior of $J_1 \cap J_2$.
It follows from $\varphi_1(t_1) = \varphi_2(t_1)$ and the construction above that $\varphi_1(t) = \varphi_2(t)$ in an interval $[t_1 - b, t_1 + b]$ around t_1, i.e.

$$[t_1 - b, t_1 + b] \subseteqq (\varphi_1 - \varphi_2)^{\circ -1}(\{0\}).$$

Then t_1 is not a boundary point which contradicts the assumption. Hence, we conclude that

$$(\varphi_1 - \varphi_2)^{\circ -1}(\{0\}) \cap (J_1 \cap J_2) = J_1 \cap J_2,$$

thus $\varphi_1(t) = \varphi_2(t)$ on $J_1 \cap J_2$.

6. Let $\varphi : J \to U$ be a maximal solution of (i) and (ii), hence φ is unique on J, and φ cannot be extended further to a unique solution on a larger set $J' \supset J$. We shall prove that the interval J is open.

Indirect proof. Assume that J is not open, and let $t_0 \in J$ be an end point of the interval. Then there exists a $b > 0$, such that φ is a unique solution in $[t_0 - b, t_0 + b]$. This means that φ is unique on

$$J' = J \cup [t_0 - b, t_0 + b] \supset J,$$

which contains points which are not in J. This is contradicting the assumption, so we conclude that every maximal solution is defined on a maximal open interval.

13 Euler-Lagrange's equations

Example 13.1 *Let $[a,b]$ be a closed and bounded interval in $\mathbb{R}$. Denote by*

$$C^1([a,b],\mathbb{R}^n)$$

the vector space of differentiable curves $x:[a,b]\to\mathbb{R}^n$ in $\mathbb{R}^n$ of class C^1. Equip $C^1([a,b],\mathbb{R}^n)$ with the norm

$$\|x\|_1 = \sup\{\|x(t)\|+\|x'(t)\| \mid t\in[a,b]\},$$

in which $\|\cdot\|$ is the maximum norm in $\mathbb{R}^n$.
For an arbitrary open set U in $\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n$, we denote by $\tilde{U}$ the subset of curves $x\in C^1([a,b],\mathbb{R}^n)$, in which $(t,x(t),x'(t))\in U$ for all $t\in[a,b]$.

1. *Show that $\tilde{U}$ is an open set in $C^1([a,b],\mathbb{R}^n)$.*

Now let U be an open set in $\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n$, considered with coordinates

$$(t,q,p)\in\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n,$$

and let $L=L(t,q,p):U\to\mathbb{R}$ be a differentiable function of class C^1.
Define the function $f:\tilde{U}\to\mathbb{R}$ by

$$f(x)=\int_a^b L(t,x(t),x'(t))\,dt \qquad \text{for } x\in\tilde{U}.$$

2. *Show that $f:\tilde{U}\to\mathbb{R}$ is differentiable in $\tilde{U}$ with the differential determined by*

$$Df(x)h=\int_a^b DL(t,x(t),x'(t))\cdot(0,h(t),h'(t))\,dt$$

for $x\in\tilde{U}$ and $h\in C^1([a,b],\mathbb{R}^n)$.

In the following, the curves $x\in\tilde{U}$ and $h\in C^1([a,b],\mathbb{R}^n)$ are kept fixed.

3. *Show that there exists an $\varepsilon>0$, such that the curve $x+\lambda h$ belongs to $\tilde{U}$, for all $\lambda\in]-\varepsilon,\varepsilon[$.*

With reference to (3), define the function $g:]-\varepsilon,\varepsilon[\to\mathbb{R}$ by

$$g(\lambda)=f(x+\lambda h) \qquad \text{for } \lambda\in]-\varepsilon,\varepsilon[.$$

4. *Show that g is differentiable at $\lambda=0$ with the differential quotient*

$$\begin{aligned} g'(0) &= \int_a^b DL(t,x(t),x'(t))\cdot(0,h(t),h'(t))\,dt \\ &= \int_a^b \left\{\sum_{i=1}^n \left(\frac{\partial L}{\partial q_i}\,h_i+\frac{\partial L}{\partial p_i}\,h_i'\right)\right\}dt. \end{aligned}$$

Here, as well as in (5), the partial derivatives of L shall be taken at the points $(t,x(t),x'(t))\in U$ and the functions h_i, h_i' at $t\in[a,b]$.

5. *Now assume that $h(a)=h(b)=0$ and that*

$$L=L(t,q,p):U\to\mathbb{R}$$

is a differentiable function of class C^2.

Using integration by parts, first show that

$$\int_a^b \frac{\partial L}{\partial p_i}\,h_i'\,dt = -\int_a^b \frac{d}{dt}\left(\frac{\partial L}{\partial p_i}\right)h_i\,dt,$$

and next that

$$g'(0) = \int_a^b \left\{ \sum_{i=1}^n \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial p_i}\right)\right) h_i \right\} dt.$$

The system of equations

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial p_i}\right) = 0, \qquad i = 1, \dots, n,$$

is called the Euler-Lagrange equations *for the above function $f = f(x)$ defined by L. It is of fundamental importance in the calculus of variations.*

6. *Show that the differentiable curve $x : [a,b] \to \mathbb{R}^n$ in $\tilde{U}$ is a stationary point of $f : \tilde{U} \to \mathbb{R}$, i.e. $Df(x) = 0$, if and only if*

$$\frac{\partial L}{\partial q_i}(t, x(t), x'(t)) = \frac{d}{dt}\left(\frac{\partial L}{\partial p_i}(t, x(t), x'(t))\right),$$

for all $i = 1, \dots, n$.

1. Let

$$x_0 \in \tilde{U} = \{x \in C^1([a,b], \mathbb{R}) \mid \forall t \in [a,b] : (t, x(t), x'(t)) \in U\}.$$

The mapping $t \mapsto (t, x_0(t), x_0'(t))$ is continuous $[a,b] \to \mathbb{R}^{2n+1}$, and $[a,b]$ is compact, hence the graph

$$A = \{(t, x_0(t), x_0'(t)) \mid t \in [a,b]\}$$

is compact.
The compact set A and the closed set $\mathbb{R}^{2n+1} \setminus U$ are disjoint, hence

$$\operatorname{dist}\left(A, \mathbb{R}^{2n+1} \setminus U\right) > 0.$$

We have for every $\varepsilon \in]0, \operatorname{dist}\left(A, \mathbb{R}^{2n+1} \setminus U\right)[$ that

$$\{x \in C^1([a,b], \mathbb{R}^n) \mid \|x - x_0\|_1 < \varepsilon\} \subset \tilde{U}.$$

This is true for every $x_0 \in \tilde{U}$ with $\varepsilon = \varepsilon(x_0) > 0$, so $\tilde{U}$ is open.

2. Let $x \in \tilde{U}$ and $h \in C^1([a,b], \mathbb{R}^n)$ with $x + h \in \tilde{U}$. Then

$$\begin{aligned} Df(x)h &= f(x+h) - f(x) + \varepsilon_1(x,h)\|h\| \\ &= \int_a^b \{L(t, x+h, x'+h') - L(t, x, x')\}\, dt + \varepsilon_1(x,h)\|h\| \\ &= \int_a^b \{DL(t, x(t), x'(t)) \cdot (0, h(t), h'(t)) + \varepsilon_2(x,h)\|h\|\}\, dt + \varepsilon_1(x,h)\|h\|. \end{aligned}$$

The interval $[a,b]$ is compact, so

$$\int_a^b \varepsilon_2(x,h)\|h\|\, dt = \varepsilon_3(x,h)\|h\|,$$

and we get by taking the limit,

$$Df(x)h = \int_a^b DL(t, x(t), x'(t)) \cdot (0, h(t), h'(t))\, dt.$$

3. There is nothing to prove for $h = 0$. If $h \neq 0$, choose ε, such that

$$0 < \varepsilon < \frac{1}{\|h\|_1} \operatorname{dist}\left(A, \mathbb{R}^{2n+1} \setminus U\right),$$

cf. (1). Then $x + \lambda h \in \tilde{U}$ for every $\lambda \in]-\varepsilon, \varepsilon[$.

4. Now, $g'(\lambda) = Df(x + \lambda h) \cdot h$, so it folows for $\lambda = 0$ from (2) that

$$g'(0) = Df(x) \cdot h = \int_a^b DL(t, x(t), x'(t)) \cdot (0, h(t), h'(t))\, dt = \int_a^b \left\{ \sum_{i=1}^n \left(\frac{\partial L}{\partial q_i} h_i + \frac{\partial L}{\partial p_i} h_i' \right) \right\} dt.$$

5. The task is almost described completely in the beginning of the example.
Since $h_i(a) = h_i(b) = 0$, we get by a partial integration that

$$\begin{aligned} \int_a^b \frac{\partial L}{\partial p_i}(t) \cdot h_i'(t)\, dt &= \left[\frac{\partial L}{\partial p_i}(t) \cdot h_i(t) \right]_a^b - \int_a^b \frac{d}{dt} \left(\frac{\partial L}{\partial p_i} \right) h_i(t)\, dt \\ &= -\int_a^b \frac{d}{dt} \left(\frac{\partial L}{\partial p_i} \right) \cdot h_i(t)\, dt. \end{aligned}$$

When this is inserted into (4), it follows that

$$g'(0) = \int_a^b \left\{ \sum_{i=1}^n \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial p_i} \right) \right) h_i \right\} dt.$$

6. If $x \in \tilde{U}$ is a stationary point of $f : \tilde{U} \to \mathbb{R}$, i.e. $Df(x) = 0$, then $g'(0) = 0$ for every $h \in C^1$. Then it follows from (5) that

$$g'(0) = \int_a^b \left\{ \sum_{i=1}^n \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial p_i}\right)\right) h_i \right\} dt = 0 \quad \text{for alle } h \in C^1.$$

Choosing in particular

$$h_i = \frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial p_i}\right), \qquad i = 1, \dots, n,$$

we see that this is only possible, if

$$(6) \qquad \frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial p_i}\right) = 0, \qquad \text{for } i = 1, \dots, n.$$

Conversely, if (6) holds, then clearly $g'(0) = 0$, and thus $Df(x) = 0$, so x is a stationary point for $f : \tilde{U} \to \mathbb{R}$.